GLUING DERIVED EQUIVALENCES TOGETHER

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Abstract. The Grothendieck construction of a diagram X of categories can be seen as a process to construct a single category $\mathrm{Gr}(X)$ by gluing categories in the diagram together. Here we formulate diagrams of categories as colax functors from a small category I to the 2-category $\Bbbk$-**Cat** of small $\Bbbk$-categories for a fixed commutative ring $\Bbbk$. In our previous paper we defined derived equivalences of those colax functors. Roughly speaking two colax functors $X, X'\colon I \to \Bbbk\text{-}\mathbf{Cat}$ are derived equivalent if there is a derived equivalence from $X(i)$ to $X'(i)$ for all objects i in I satisfying some "I-equivariance" conditions. In this paper we glue the derived equivalences between $X(i)$ and $X'(i)$ together to obtain a derived equivalence between Grothendieck constructions $\mathrm{Gr}(X)$ and $\mathrm{Gr}(X')$, which shows that if colax functors are derived equivalent, then so are their Grothendieck constructions. This generalizes and well formulates the fact that if two $\Bbbk$-categories with a G-action for a group G are "G-equivariantly" derived equivalent, then their orbit categories are derived equivalent. As an easy application we see by a unified proof that if two $\Bbbk$-algebras A and A' are derived equivalent, then so are the path categories AQ and $A'Q$ for any quiver Q; so are the incidence categories AS and $A'S$ for any poset S; and so are the monoid algebras AG and $A'G$ for any monoid G. Also we will give examples of gluing of many smaller derived equivalences together to have a larger derived equivalence.

Keywords: Grothendieck constructions, 2-categories, lax functors, colax functors, pseudofunctors, derived equivalences

1. Introduction

Under the preparations in [6] we complete our project of the title in this paper. We fix a small category I and a commutative ring $\Bbbk$ and denote by $\Bbbk$-**Cat** (resp. $\Bbbk$-**Ab**, $\Bbbk$-**Tri**) the 2-category of small $\Bbbk$-categories (resp. abelian $\Bbbk$-categories, triangulated $\Bbbk$-categories). For a $\Bbbk$-category $\mathcal{C}$ a (right) $\mathcal{C}$-*module* is a contravariant functor from $\mathcal{C}$ to the category $\mathrm{Mod}\,\Bbbk$ of $\Bbbk$-modules, and we denote by $\mathrm{Mod}\,\mathcal{C}$ (resp. $\mathrm{Prj}\,\mathcal{C}$, $\mathrm{prj}\,\mathcal{C}$) the category of $\mathcal{C}$-modules (resp. projective $\mathcal{C}$-modules, finitely generated projective $\mathcal{C}$-modules). When we deal with derived equivalences, we usually assume that $\Bbbk$ is a field because Keller's theorem in [11] or [12] on derived equivalences of categories requires that the $\Bbbk$-categories in consideration are $\Bbbk$-flat or $\Bbbk$-projective.

A $\Bbbk$-category $\mathcal{C}$ with an action of a group G have been well investigated in connection with a so-called covering technique in representation theory of algebras (see e.g., [8]). The orbit category $\mathcal{C}/G$ and the canonical functor $\mathcal{C} \to \mathcal{C}/G$ are naturally constructed from these data, and one studied relationships between $\mathrm{Mod}\,\mathcal{C}$ and $\mathrm{Mod}\,\mathcal{C}/G$.

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We brought this point of view to the derived equivalence classification problem of algebras in [1], and a main tool obtained there was fully used in the derived equivalence classifications in [2, 3]. The main tool was extended in [4] in the following form:

Theorem 1.1. *Let G be a group acting on categories $\mathcal{C}$ and $\mathcal{C}'$. Assume that $\mathcal{C}$ is $\Bbbk$-flat and that the following condition is satisfied:*

- $(*)$ *There exists a G-stable tilting subcategory E of $\mathcal{K}^{\mathrm{b}}(\operatorname{prj}\mathcal{C})$ such that there is a G-equivariant equivalence $\mathcal{C}' \to E$.*

Then the orbit categories $\mathcal{C}/G$ and $\mathcal{C}'/G$ are derived equivalent.

(In the above, $\mathcal{C}$ is called $\Bbbk$-*flat* if all morphism spaces are flat $\Bbbk$-modules, and E is said to be G-*stable* if the set of objects in E is stable under the G-action on $\mathcal{K}^{\mathrm{b}}(\operatorname{prj}\mathcal{C})$ induced from that on $\mathcal{C}$.) Observe that if we regard G as a category with a single object $*$, then a G-action on a category $\mathcal{C}$ is nothing but a functor $X : G \to \Bbbk$-**Cat** with $X(*) = \mathcal{C}$; and the orbit category $\mathcal{C}/G$ coincides with (the $\Bbbk$-linear version of) the Grothendieck construction $\operatorname{Gr}(X)$ of X defined in [10].

The purpose of this paper is to generalize this theorem to an arbitrary category I and to any *colax functors*[1] $X, X' : I \to \Bbbk$-**Cat** (roughly speaking a colax functor X is a family $(X(i))_{i \in I_0}$ of $\Bbbk$-categories indexed by the objects i of I with an action of I, the precise definition is given in Definition 2.1). Recall that if $\mathcal{C}$ is a category with an action of a group G, then the module category $\operatorname{Mod}\mathcal{C}$ (resp. the derived category $\mathcal{D}(\operatorname{Mod}\mathcal{C})$) has the induced G-action; thus both of them are again categories with G-actions. Hence for a colax functor X the "module category" $\operatorname{Mod} X$ (resp. the "derived category" $\mathcal{D}(\operatorname{Mod} X)$) should again be a family of categories with an I-action, i.e., a colax functor from I to $\Bbbk$-**Ab** (resp. to $\Bbbk$-**Tri**). In addition, we need a notion of equivalences between colax functors for two purposes:

(a) to generalize the statement $(*)$; and
(b) to define a derived equivalence of colax functors X, X' by an existence of an equivalence between colax functors $\mathcal{D}(\operatorname{Mod} X)$ and $\mathcal{D}(\operatorname{Mod} X')$.

To define equivalences of objects we need notions of 1-morphisms and 2-morphisms, thus we need a 2-categorical structure on the collection of colax functors, i.e., it is needed to define a 2-category $\overleftarrow{\operatorname{Colax}}(I, \mathbf{C})$ of all colax functors from I to a 2-category $\mathbf{C}$, which can be used for both (a) and (b) above. Having these things in mind we see that to generalize the theorem above we have to solve the following problems:

(1) Define the "module category" of a colax functor again as a colax functor.
(2) Define the "derived category" of a colax functor as a colax functor.
(3) Give a natural definition of an equivalence between colax functors using 2-morphisms of the 2-category of colax functors.
(4) Give a condition on a 1-morphism between colax functors to be an equivalence.
(5) Give a natural definition of a derived equivalence between colax functors by the equivalence (defined in (3)) of their "derived categories" defined in (2).

[1] In [6] we called them *oplax* functors. There are two versions of Grothendieck constructions: (1) for contravariant lax functors and (2) for covariant colax functors. Since skew group algebras are formulated as the second version we deal with colax functors here. See [4, Example 2.12].

(6) Characterize the existence of derived equivalences of colax functors by tilting subcategories, which turns out to be a generalization of Rickard's Morita theorem for colax functors.

(7) Induce a derived equivalence of Grothendieck constructions of colax functors from the existence of tilting subcategories, which will be a generalization of the theorem above.

In our previous paper [6] we have solved the problems (1) – (6) and made clear the meaning of the condition $(*)$ in the setting of colax functors. In this paper we solve the problem (7), and in addition we give a unified way to solve (1) and (2) using the following general statement on compositions with pseudofunctors (cf. Gordon–Power–Street [9, Subsection 5.6]):

Theorem (Theorem 6.5)**.** *Let* $\mathbf{B}, \mathbf{C}$ *and* $\mathbf{D}$ *be* 2*-categories and* $V\colon \mathbf{C} \to \mathbf{D}$ *a pseudofunctor. Then the obvious correspondence* (*see subsection 9.1 for details*)

$$\overleftarrow{\mathrm{Colax}}(\mathbf{B}, V)\colon \overleftarrow{\mathrm{Colax}}(\mathbf{B}, \mathbf{C}) \to \overleftarrow{\mathrm{Colax}}(\mathbf{B}, \mathbf{D})$$

turns out to be a pseudofunctor.

The solutions of (1) and (2) use the correspondence on objects given by the pseudofunctor $\overleftarrow{\mathrm{Colax}}(\mathbf{B}, V)$. The correspondence on 1-morphisms is needed also to solve (7). The following is our main result (see Definition 7.4 for definitions):

Theorem (Theorem 8.2)**.** *Let* $X, X' \in \overleftarrow{\mathrm{Colax}}(I, \Bbbk\text{-}\mathbf{Cat})$*. Assume that* X *is* $\Bbbk$*-flat and that there exists a tilting colax functor* $\mathcal{T}$ *for* X *such that* $\mathcal{T}$ *and* X' *are equivalent in* $\overleftarrow{\mathrm{Colax}}(I, \Bbbk\text{-}\mathbf{Cat})$*. Then* $\mathrm{Gr}(X)$ *and* $\mathrm{Gr}(X')$ *are derived equivalent.*

Note that there is an easier way (Lemma 7.1, a solution of (4)) to verify that $\mathcal{T}$ and X' are equivalent in $\overleftarrow{\mathrm{Colax}}(I, \Bbbk\text{-}\mathbf{Cat})$ in the above. As an easy application, the theorem above gives a unified proof of the following.

Theorem (Theorem 8.5)**.** *Assume that* $\Bbbk$ *is a field and that* $\Bbbk$*-algebras* A *and* A' *are derived equivalent. Then the following pairs are derived equivalent as well:*

(1) *path-categories* AQ *and* $A'Q$ *for any quiver* Q*;*
(2) *incidence categories* AS *and* $A'S$ *for any poset* S*; and*
(3) *monoid algebras* AG *and* $A'G$ *for any monoid* G*.*

Theorem 8.2 can be used to glue many derived equivalences together as shown in Example 8.6.

The paper is organized as follows. In section 2 we recall the definition of the 2-category $\overleftarrow{\mathrm{Colax}}(I, \mathbf{C})$ of colax functors from a category I to a 2-category $\mathbf{C}$. In section 3 we first define a diagonal 2-functor $\Delta\colon \Bbbk\text{-}\mathbf{Cat} \to \overleftarrow{\mathrm{Colax}}(I, \Bbbk\text{-}\mathbf{Cat})$ in an obvious way, and introduce a notion of I-coverings $(F, \psi)\colon X \to \Delta(\mathcal{C})$ for a colax functor $X \in \overleftarrow{\mathrm{Colax}}(I, \Bbbk\text{-}\mathbf{Cat})_0$ and $\mathcal{C} \in \Bbbk\text{-}\mathbf{Cat}_0$ (the subscript 0 stands for objects) as a generalization of G-coverings for a group G. In section 4 we define a $\Bbbk$-linear version of Grothendieck construction as a 2-functor $\mathrm{Gr}\colon \overleftarrow{\mathrm{Colax}}(I, \Bbbk\text{-}\mathbf{Cat}) \to \Bbbk\text{-}\mathbf{Cat}$ and introduce the canonical morphism $(P, \phi)\colon X \to \Delta(\mathrm{Gr}(X))$. In section 5 we will show that the

Grothendieck construction is a strict left adjoint to the diagonal 2-functor with a unit given by the family of canonical morphisms, in particular, this shows that the canonical morphism $(P, \phi)\colon X \to \Delta(\mathrm{Gr}(X))$ is an I-covering and any other I-covering $X \to \Delta(\mathcal{C})$ is given as the composite of this followed by $\Delta(H)$ for an equivalence $H\colon \mathrm{Gr}(X) \to \mathcal{C}$. This will be used in the proof of the main result. In section 6 we redefine the module colax functor $\mathrm{Mod}\, X\colon I \to \Bbbk\text{-}\mathbf{Ab}$ and its derived colax functor $\mathcal{D}(\mathrm{Mod}\, X)\colon I \to \Bbbk\text{-}\mathbf{Tri}$ for a colax functor $X \in \overleftarrow{\mathrm{Colax}}(I, \Bbbk\text{-}\mathbf{Cat})_0$ by using Theorem 6.5. In addition, we also define $\mathcal{K}^{\mathrm{b}}(\mathrm{prj}\, X)$ for $X \in \overleftarrow{\mathrm{Colax}}(I, \Bbbk\text{-}\mathbf{Cat})_0$ and show that this construction preserves I-precoverings, which is also used in the proof of the main result. It is obvious that the definitions given here coincide with those given in our previous paper [6]. In section 7 we recall the definition of derived equivalences of colax functors in $\overleftarrow{\mathrm{Colax}}(I, \Bbbk\text{-}\mathbf{Cat})$ and the theorem characterizing the derived equivalence by tilting colax functors (Theorem 7.5). In section 8 we give a proof of Theorem 8.2, and give some applications including an example of gluing of pieces of derived equivalences together to have a larger one. In the last section we give a proof of Theorem 6.5.

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2. Preliminaries

In this section we recall the definition of the 2-category of colax functors from I to a 2-category from [6] (see also Tamaki [15]).

Definition 2.1. Let $\mathbf{C}$ be a 2-category. A *colax functor* (or an *oplax* functor) from I to $\mathbf{C}$ is a triple (X, η, θ) of data:

- a quiver morphism $X\colon I \to \mathbf{C}$, where I and $\mathbf{C}$ are regarded as quivers by forgetting additional data such as 2-morphisms or compositions;

- a family $\eta := (\eta_i)_{i\in I_0}$ of 2-morphisms $\eta_i \colon X(\mathbb{1}_i) \Rightarrow \mathbb{1}_{X(i)}$ in $\mathbf{C}$ indexed by $i \in I_0$; and
- a family $\theta := (\theta_{b,a})_{(b,a)}$ of 2-morphisms $\theta_{b,a} \colon X(ba) \Rightarrow X(b)X(a)$ in $\mathbf{C}$ indexed by $(b,a) \in \mathrm{com}(I) := \{(b,a) \in I_1 \times I_1 \mid ba \text{ is defined}\}$

satisfying the axioms:

(a) For each $a \colon i \to j$ in I the following are commutative:

$$\begin{tikzcd} X(a\mathbb{1}_i) \arrow[r, Rightarrow, "\theta_{a,1_i}"] \arrow[dr, equal] & X(a)X(\mathbb{1}_i) \arrow[d, Rightarrow, "X(a)\eta_i"] \\ & X(a)\mathbb{1}_{X(i)} \end{tikzcd} \quad \text{and} \quad \begin{tikzcd} X(\mathbb{1}_j a) \arrow[r, Rightarrow, "\theta_{1_j,a}"] \arrow[dr, equal] & X(\mathbb{1}_j)X(a) \arrow[d, Rightarrow, "\eta_j X(a)"] \\ & \mathbb{1}_{X(j)}X(a) \end{tikzcd} \quad ; \text{ and}$$

(b) For each $i \xrightarrow{a} j \xrightarrow{b} k \xrightarrow{c} l$ in I the following is commutative:

$$\begin{tikzcd} X(cba) \arrow[r, Rightarrow, "\theta_{c,ba}"] \arrow[d, Rightarrow, "\theta_{cb,a}"'] & X(c)X(ba) \arrow[d, Rightarrow, "X(c)\theta_{b,a}"] \\ X(cb)X(a) \arrow[r, Rightarrow, "\theta_{c,b}X(a)"'] & X(c)X(b)X(a). \end{tikzcd}$$

Definition 2.2. Let $\mathbf{C}$ be a 2-category and $X = (X, \eta, \theta)$, $X' = (X', \eta', \theta')$ be colax functors from I to $\mathbf{C}$. A 1-*morphism* (called a *left transformation*) from X to X' is a pair (F, ψ) of data

- a family $F := (F(i))_{i\in I_0}$ of 1-morphisms $F(i) \colon X(i) \to X'(i)$ in $\mathbf{C}$; and
- a family $\psi := (\psi(a))_{a\in I_1}$ of 2-morphisms $\psi(a) \colon X'(a)F(i) \Rightarrow F(j)X(a)$

$$\begin{tikzcd} X(i) \arrow[r, "F(i)"] \arrow[d, "X(a)"'] & X'(i) \arrow[d, "X'(a)"] \arrow[dl, Rightarrow, "\psi(a)"'] \\ X(j) \arrow[r, "F(j)"'] & X'(j) \end{tikzcd}$$

in $\mathbf{C}$ indexed by $a \colon i \to j$ in I_1

satisfying the axioms

(a) For each $i \in I_0$ the following is commutative:

$$\begin{tikzcd} X'(\mathbb{1}_i)F(i) \arrow[r, Rightarrow, "\psi(\mathbf{1}_i)"] \arrow[d, Rightarrow, "\eta'_i F(i)"'] & F(i)X(\mathbb{1}_i) \arrow[d, Rightarrow, "F(i)\eta_i"] \\ \mathbb{1}_{X'(i)}F(i) \arrow[r, equal] & F(i)\mathbb{1}_{X(i)} \end{tikzcd} \quad ; \text{ and}$$

(b) For each $i \xrightarrow{a} j \xrightarrow{b} k$ in I the following is commutative:

$$\begin{tikzcd}
X'(ba)F(i) \arrow[r, Rightarrow, "\theta'_{b,a}F(i)"] \arrow[d, Rightarrow, "\psi(ba)"'] & X'(b)X'(a)F(i) \arrow[r, Rightarrow, "X'(b)\psi(a)"] & X'(b)F(j)X(a) \arrow[d, Rightarrow, "\psi(b)X(a)"] \\
F(k)X(ba) \arrow[rr, Rightarrow, "F(k)\,\theta_{b,a}"'] & & F(k)X(b)X(a).
\end{tikzcd}$$

A 1-morphism $(F, \psi)\colon X \to X'$ is said to be I-*equivariant* if $\psi(a)$ is a 2-isomorphism in $\mathbf{C}$ for all $a \in I_1$.

Definition 2.3. Let $\mathbf{C}$ be a 2-category, $X = (X, \eta, \theta)$, $X' = (X', \eta', \theta')$ be colax functors from I to $\mathbf{C}$, and (F, ψ), (F', ψ') 1-morphisms from X to X'. A 2-*morphism* from (F, ψ) to (F', ψ') is a family $\zeta = (\zeta(i))_{i\in I_0}$ of 2-morphisms $\zeta(i)\colon F(i) \Rightarrow F'(i)$ in $\mathbf{C}$ indexed by $i \in I_0$ such that the following is commutative for all $a\colon i \to j$ in I:

$$\begin{tikzcd}
X'(a)F(i) \arrow[r, Rightarrow, "X'(a)\zeta(i)"] \arrow[d, Rightarrow, "\psi(a)"'] & X'(a)F'(i) \arrow[d, Rightarrow, "\psi'(a)"] \\
F(j)X(a) \arrow[r, Rightarrow, "\zeta(j)X(a)"] & F'(j)X(a).
\end{tikzcd}$$

Definition 2.4. Let $\mathbf{C}$ be a 2-category, $X = (X, \eta, \theta)$, $X' = (X', \eta', \theta')$ and $X'' = (X'', \eta'', \theta'')$ be colax functors from I to $\mathbf{C}$, and let $(F, \psi)\colon X \to X'$, $(F', \psi')\colon X' \to X''$ be 1-morphisms. Then the composite $(F', \psi')(F, \psi)$ of (F, ψ) and (F', ψ') is a 1-morphism from X to X'' defined by

$$(F', \psi')(F, \psi) := (F'F, \psi' \circ \psi),$$

where $F'F := ((F'(i)F(i))_{i\in I_0}$ and for each $a\colon i \to j$ in I, $(\psi' \circ \psi)(a) := F'(j)\psi(a) \circ \psi'(a)F(i)$ is the pasting of the diagram

$$\begin{tikzcd}
X(i) \arrow[r, "F(i)"] \arrow[d, "X(a)"'] & X'(i) \arrow[r, "F'(i)"] \arrow[d, "X'(a)"] \arrow[dl, phantom, "\swArrow\psi(a)"] & X''(i) \arrow[d, "X''(a)"] \arrow[dl, phantom, "\swArrow\psi'(a)"] \\
X(j) \arrow[r, "F(j)"'] & X'(j) \arrow[r, "F'(j)"'] & X''(j).
\end{tikzcd}$$

The following is straightforward to verify.

Proposition 2.5. *Let* $\mathbf{C}$ *be a* 2*-category. Then colax functors* $I \to \mathbf{C}$*,* 1*-morphisms between them, and* 2*-morphisms between* 1*-morphisms* (*defined above*) *define a* 2*-category, which we denote by* $\overleftarrow{\mathrm{Colax}}(I, \mathbf{C})$.

Notation 2.6. Let $\mathbf{C}$ be a 2-category. Then we denote by $\mathbf{C}^{\mathrm{op}}$ (resp. $\mathbf{C}^{\mathrm{co}}$) the 2-category obtained from $\mathbf{C}$ by reversing the 1-morphisms (resp. the 2-morphisms), and we set $\mathbf{C}^{\mathrm{coop}} := (\mathbf{C}^{\mathrm{co}})^{\mathrm{op}} = (\mathbf{C}^{\mathrm{op}})^{\mathrm{co}}$.

3. I-coverings

In this section we introduce the notion of I-coverings that is a generalization of that of G-coverings for a group G introduced in [4], which was obtained by generalizing the notion of Galois coverings introduced by Gabriel in [8]. This will be used in the proof of our main theorem.

Definition 3.1. We define a 2-functor $\Delta\colon \Bbbk\text{-}\mathbf{Cat} \to \overleftarrow{\mathrm{Colax}}(I, \Bbbk\text{-}\mathbf{Cat})$ as follows, which is called the *diagonal* 2-functor:

- Let $\mathcal{C} \in \Bbbk\text{-}\mathbf{Cat}$. Then $\Delta(\mathcal{C})$ is defined to be a functor sending each morphism $a\colon i \to j$ in I to $\mathbb{1}_{\mathcal{C}}\colon \mathcal{C} \to \mathcal{C}$.
- Let $E\colon \mathcal{C} \to \mathcal{C}'$ be a 1-morphism in $\Bbbk\text{-}\mathbf{Cat}$. Then $\Delta(E)\colon \Delta(\mathcal{C}) \to \Delta(\mathcal{C}')$ is a 1-morphism (F, ψ) in $\overleftarrow{\mathrm{Colax}}(I, \Bbbk\text{-}\mathbf{Cat})$ defined by $F(i) := E$ and $\psi(a) := \mathbb{1}_E$ for all $i \in I_0$ and all $a \in I_1$:

$$\begin{tikzcd} \mathcal{C} \arrow[r, "E"] \arrow[d, "\mathbf{1}_{\mathcal{C}}"'] & \mathcal{C}' \arrow[d, "\mathbf{1}_{\mathcal{C}'}"] \arrow[dl, Rightarrow, "\mathbf{1}_E"'] \\ \mathcal{C} \arrow[r, "E"] & \mathcal{C}'. \end{tikzcd}$$

- Let $E, E'\colon \mathcal{C} \to \mathcal{C}'$ be 1-morphisms in $\Bbbk\text{-}\mathbf{Cat}$, and $\alpha\colon E \Rightarrow E'$ a 2-morphism in $\Bbbk\text{-}\mathbf{Cat}$. Then $\Delta(\alpha)\colon \Delta(E) \Rightarrow \Delta(E')$ is a 2-morphism in $\overleftarrow{\mathrm{Colax}}(I, \Bbbk\text{-}\mathbf{Cat})$ defined by $\Delta(\alpha) := (\alpha)_{i\in I_0}$.

Remark 3.2. Let $\mathbf{C}$ be a 2-category, $X = (X, \eta, \theta) \in \overleftarrow{\mathrm{Colax}}(I, \mathbf{C})$, and $C \in \mathbf{C}_0$. Further let

- F be a family of 1-morphisms $F(i)\colon X(i) \to C$ in $\mathbf{C}$ indexed by $i \in I_0$; and
- ψ be a family 2-morphisms $\psi(a)\colon F(i) \Rightarrow F(j)X(a)$ indexed by $a\colon i \to j$ in I:

$$\begin{tikzcd} X(i) \arrow[r, "F(i)"] \arrow[d, "X(a)"'] & C \arrow[d, equal] \arrow[dl, Rightarrow] \\ X(j) \arrow[r, "F(j)"'] & C \end{tikzcd}$$

Then (F, ψ) is in $\overleftarrow{\mathrm{Colax}}(I, \mathbf{C})(X, \Delta(C))$ if and only if the following hold.

(a) For each $i \in I_0$ the following is commutative:

$$\begin{tikzcd} F(i) \arrow[r, Rightarrow, "\psi(\mathbf{1}_i)"] \arrow[dr, equal] & F(i)X(\mathbb{1}_i) \arrow[d, Rightarrow, "F(i)\eta_i"] \\ & F(i)\mathbb{1}_{X(i)} \end{tikzcd} \quad ; \text{ and}$$

(b) For each $i \xrightarrow{a} j \xrightarrow{b} k$ in I the following is commutative:

$$\begin{tikzcd}
F(i) \arrow[r, Rightarrow, "\psi(a)"] \arrow[d, Rightarrow, "\psi(ba)"'] & F(j)X(a) \arrow[d, Rightarrow, "\psi(b)X(a)"] \\
F(k)X(ba) \arrow[r, Rightarrow, "F(k)\,\theta_{b,a}"'] & F(k)X(b)X(a).
\end{tikzcd}$$

Definition 3.3. Let $\mathcal{C} \in \Bbbk\text{-}\mathbf{Cat}$ and $(F, \psi)\colon X \to \Delta(\mathcal{C})$ be in $\overleftarrow{\mathrm{Colax}}(I, \Bbbk\text{-}\mathbf{Cat})$. Then

(1) (F, ψ) is called an *I-precovering* (of $\mathcal{C}$) if the homomorphism

$$(F, \psi)^{(1)}_{x,y}\colon \bigoplus_{a \in I(i,j)} X(j)(X(a)x, y) \to \mathcal{C}(F(i)x, F(j)y)$$

of $\Bbbk$-modules defined by $(f_a\colon X(a)x \to y)_{a \in I(i,j)} \mapsto \sum_{a \in I(i,j)} F(j)(f_a) \circ \psi(a)(x)$ is an isomorphism for all $i, j \in I_0$ and all $x \in X(i)_0$, $y \in X(j)_0$.

(2) (F, ψ) is called an *I-covering* if it is an I-precovering and is *dense*, i.e., for each $c \in \mathcal{C}_0$ there exists an $i \in I_0$ and $x \in X(i)_0$ such that $F(i)(x)$ is isomorphic to c in $\mathcal{C}$.

4. Grothendieck constructions

In this section we define a 2-functor $\mathrm{Gr}\colon \overleftarrow{\mathrm{Colax}}(I, \Bbbk\text{-}\mathbf{Cat}) \to \Bbbk\text{-}\mathbf{Cat}$ whose correspondence on objects is a $\Bbbk$-linear version of (the opposite version of) the original Grothendieck construction (cf. [15]).

Definition 4.1. We define a 2-functor $\mathrm{Gr}\colon \overleftarrow{\mathrm{Colax}}(I, \Bbbk\text{-}\mathbf{Cat}) \to \Bbbk\text{-}\mathbf{Cat}$, which is called the *Grothendieck construction*.

On objects. Let $X = (X, \eta, \theta) \in \overleftarrow{\mathrm{Colax}}(I, \Bbbk\text{-}\mathbf{Cat})_0$. Then $\mathrm{Gr}(X) \in \Bbbk\text{-}\mathbf{Cat}_0$ is defined as follows.

- $\mathrm{Gr}(X)_0 := \bigcup_{i \in I_0} \{i\} \times X(i)_0 = \{{}_ix := (i, x) \mid i \in I_0, x \in X(i)_0\}$.
- For each ${}_ix, {}_jy \in \mathrm{Gr}(X)_0$, we set

$$\mathrm{Gr}(X)({}_ix, {}_jy) := \bigoplus_{a \in I(i,j)} X(j)(X(a)x, y).$$

- For each ${}_ix, {}_jy, {}_kz \in \mathrm{Gr}(X)_0$ and each $f = (f_a)_{a \in I(i,j)} \in \mathrm{Gr}(X)({}_ix, {}_jy)$, $g = (g_b)_{b \in I(j,k)} \in \mathrm{Gr}(X)({}_jy, {}_kz)$, we set

$$g \circ f := \left(\sum_{\substack{a \in I(i,j) \\ b \in I(j,k) \\ c = ba}} g_b \circ X(b)f_a \circ \theta_{b,a}x \right)_{c \in I(i,k)},$$

where each summand is the composite of

$$X(ba)x \xrightarrow{\theta_{b,a}x} X(b)X(a)x \xrightarrow{X(b)f_a} X(b)y \xrightarrow{g_b} z.$$

- For each ${}_ix \in \mathrm{Gr}(X)_0$ the identity $\mathbb{1}_{{}_ix}$ is given by

$$\mathbb{1}_{{}_ix} = (\delta_{a,\mathbf{1}_i}\eta_i\, x)_{a\in I(i,i)} \in \bigoplus_{a\in I(i,i)} X(i)(X(a)x, x),$$

where δ is the Kronecker delta[2].

On 1-morphisms. Let $X = (X, \eta, \theta), X' = (X', \eta', \theta')$ be objects of $\overleftarrow{\mathrm{Colax}}(I, \Bbbk\text{-}\mathbf{Cat})$ and $(F, \psi)\colon X \to X'$ a 1-morphism in $\overleftarrow{\mathrm{Colax}}(I, \Bbbk\text{-}\mathbf{Cat})$. Then a 1-morphism

$$\mathrm{Gr}(F, \psi)\colon \mathrm{Gr}(X) \to \mathrm{Gr}(X')$$

in $\Bbbk$-**Cat** is defined as follows.

- For each ${}_ix \in \mathrm{Gr}(X)_0$, $\mathrm{Gr}(F, \psi)({}_ix) := {}_i(F(i)x)$.
- For each ${}_ix, {}_jy \in \mathrm{Gr}(X)_0$ and each $f = (f_a)_{a\in I(i,j)} \in \mathrm{Gr}(X)({}_ix, {}_jy)$, we set $\mathrm{Gr}(F, \psi)(f) := (F(j)f_a \circ \psi(a)x)_{a\in I(i,j)}$, where each entry is the composite of

$$X'(a)F(i)x \xrightarrow{\psi(a)x} F(j)X(a)x \xrightarrow{F(j)f_a} F(j)y.$$

On 2-morphisms. Let $X = (X, \eta, \theta), X' = (X', \eta', \theta')$ be objects of $\overleftarrow{\mathrm{Colax}}(I, \Bbbk\text{-}\mathbf{Cat})$ and $(F, \psi), (F', \psi')\colon X \to X'$ 1-morphisms in $\overleftarrow{\mathrm{Colax}}(I, \Bbbk\text{-}\mathbf{Cat})$, and let $\zeta\colon (F, \psi) \Rightarrow (F', \psi')$ be a 2-morphism in $\overleftarrow{\mathrm{Colax}}(I, \Bbbk\text{-}\mathbf{Cat})$. Then a 2-morphism

$$\mathrm{Gr}(\zeta)\colon \mathrm{Gr}(F, \psi) \Rightarrow \mathrm{Gr}(F', \psi')$$

in $\Bbbk$-**Cat** is defined by

$$\mathrm{Gr}(\zeta)_{{}_ix} := (\delta_{a,\mathbf{1}_i}\zeta(i)x)_{a\in I(i,i)}\colon {}_i(F(i)x) \to {}_i(F'(i)x)$$

in $\mathrm{Gr}(X')$ for each ${}_ix \in \mathrm{Gr}(X)_0$.

Example 4.2. Let A be a $\Bbbk$-algebra regarded as a $\Bbbk$-category with a single object. Then $A \in \Bbbk\text{-}\mathbf{Cat}_0$. Consider the functor $X := \Delta(A)\colon I \to \Bbbk\text{-}\mathbf{Cat}$. Then it is straightforward to verify the following.

(1) If I is a free category defined by the quiver $1 \to 2$, then $\mathrm{Gr}(X)$ is isomorphic to the triangular algebra $\begin{bmatrix} A & 0 \\ A & A \end{bmatrix}$.
(2) If I is a free category defined by a quiver Q, then $\mathrm{Gr}(X)$ is isomorphic to the path-category AQ of Q over A.
(3) If I is a poset S, then $\mathrm{Gr}(X)$ is isomorphic to the incidence category AS of S over A.
(4) If I is a monoid G, then $\mathrm{Gr}(X)$ is isomorphic to the monoid algebra[3] AG of G over A.

In (3) above, AS is defined to be the factor category of the path-category AQ modulo the ideal generated by the full commutativity relations in Q, where Q is the Hasse diagram of S regarded as a quiver by drawing an arrow $x \to y$ if $x \leq y$ in Q. If S is a finite poset, then AS is identified with the usual incidence algebra.

[2]This is used to mean that the a-th component is $\eta_i\, x$ if $a = \mathbb{1}_i$, and 0 otherwise.
[3]Since AG has the identity $1_A 1_G$, this is regarded as a category with a single object.

See [7] for further examples of the Grothendieck constructions of functors, in which the examples (2) and (3) above are unified and generalized.

Definition 4.3. Let $X \in \overleftarrow{\mathrm{Colax}}(I, \Bbbk\text{-}\mathbf{Cat})$. We define a left transformation $(P_X, \phi_X) := (P, \phi)\colon X \to \Delta(\mathrm{Gr}(X))$ (called the *canonical morphism*) as follows.

- For each $i \in I_0$, the functor $P(i)\colon X(i) \to \mathrm{Gr}(X)$ is defined by

$$\begin{cases} P(i)x := {}_ix \\ P(i)f := (\delta_{a,\mathbf{1}_i} f \circ (\eta_i\, x))_{a \in I(i,i)} \colon {}_ix \to {}_iy \text{ in } \mathrm{Gr}(X) \end{cases}$$

  for all $f\colon x \to y$ in $X(i)$.
- For each $a\colon i \to j$ in I, the natural transformation $\phi(a)\colon P(i) \Rightarrow P(j)X(a)$

$$\begin{tikzcd} X(i) \arrow[r, "P(i)"] \arrow[d, "X(a)"'] & \mathrm{Gr}(X) \arrow[d, equal] \arrow[dl, Rightarrow, "\phi(a)"'] \\ X(j) \arrow[r, "P(j)"'] & \mathrm{Gr}(X) \end{tikzcd}$$

  is defined by $\phi(a)x := (\delta_{b,a} \mathbb{1}_{X(a)x})_{b \in I(i,j)}$ for all $x \in X(i)_0$.

Lemma 4.4. *The (P, ϕ) defined above is a 1-morphism in $\overleftarrow{\mathrm{Colax}}(I, \Bbbk\text{-}\mathbf{Cat})$.*

Proof. This is straightforward by using Remark 3.2. □

Proposition 4.5. *Let $X \in \overleftarrow{\mathrm{Colax}}(I, \Bbbk\text{-}\mathbf{Cat})$. Then the canonical morphism $(P, \phi)\colon X \to \Delta(\mathrm{Gr}(X))$ is an I-covering. More precisely, the morphism*

$$(P, \phi)^{(1)}_{x,y} \colon \bigoplus_{a \in I(i,j)} X(j)(X(a)x, y) \to \mathrm{Gr}(X)(P(i)x, P(j)y)$$

is the identity for all $i, j \in I_0$ and all $x \in X(i)_0$, $y \in X(j)_0$.

Proof. By the definitions of $\mathrm{Gr}(X)_0$ and of P it is obvious that (P, ϕ) is dense. Let $i, j \in I_0$ and $x \in X(i)$, $y \in X(j)$. We only have to show that

$$(P, \phi)^{(1)}_{x,y} \colon \bigoplus_{a \in I(i,j)} X(j)(X(a)x, y) \to \mathrm{Gr}(X)(P(i)x, P(j)y)$$

is the identity. Let $f = (f_a)_{a\in I(i,j)} \in \bigoplus_{a\in I(i,j)} X(j)(X(a)x, y)$. Then

$$\begin{aligned}
(P,\phi)^{(1)}_{x,y}(f) &= \sum_{a\in I(i,j)} P(j)(f_a) \circ \phi(a)x \\
&= \sum_{a\in I(i,j)} (\delta_{b,\mathbf{1}_j} f_a \circ (\eta_j\, x))_{b\in I(j,j)} \circ (\delta_{c,a} \mathbb{1}_{X(a)x})_{c\in I(i,j)} \\
&= \sum_{a\in I(i,j)} \left(\sum_{\substack{b\in I(j,j)\\ c\in I(i,j)\\ d=bc}} \delta_{b,\mathbf{1}_j} f_a \circ (\eta_j\, x) \circ \delta_{c,a} \mathbb{1}_{X(b)X(a)x} \circ \theta_{b,c} x \right)_{d\in I(i,j)} \\
&= \sum_{a\in I(i,j)} \left(\delta_{d,a} f_a \circ (\eta_j\, x) \circ \mathbb{1}_{X(\mathbf{1}_j)X(a)x} \circ \theta_{\mathbf{1}_j,a} x\right)_{d\in I(i,j)} \\
&= (f_a \circ (\eta_j\, x) \circ \theta_{\mathbf{1}_j,a} x)_{a\in I(i,j)} = (f_a)_{a\in I(i,j)} \\
&= f,
\end{aligned}$$

as required. $\square$

Lemma 4.6. *Let $X \in \overleftarrow{\mathrm{Colax}}(I, \Bbbk\text{-}\mathbf{Cat})_0$ and $H\colon \mathrm{Gr}(X) \to \mathcal{C}$ be in $\Bbbk$-$\mathbf{Cat}$ and consider the composite 1-morphism $(F,\psi)\colon X \xrightarrow{(P,\phi)} \Delta(\mathrm{Gr}(X)) \xrightarrow{\Delta(H)} \Delta(\mathcal{C})$. Then (F,ψ) is an I-covering if and only if H is an equivalence.*

Proof. Obviously (F,ψ) is dense if and only if so is H. Further for each $i,j \in I_0$, $x \in X(i)$ and $y \in X(j)$, $(F,\psi)^{(1)}_{x,y}$ is an isomorphism if and only if so is $H_{{}_ix,{}_jy}$ because we have a commutative diagram

$$\begin{tikzcd}
\bigoplus_{a\in I(i,j)} X(j)(X(a)x, y) \arrow[r, "(F,\psi)^{(1)}_{x,y}"] \arrow[d, equal, "(P,\phi)^{(1)}_{x,y}"'] & \mathcal{C}(F(i)x, F(j)y) \\
\mathrm{Gr}(X)({}_ix, {}_jy) \arrow[ur, "H_{{}_ix,{}_jy}"'] &
\end{tikzcd}$$

by Proposition 4.5. $\square$

5. Adjoints

In this section we will show that the Grothendieck construction is a strict left adjoint to the diagonal 2-functor, and that I-coverings are essentially given by the unit of the adjunction.

Definition 5.1. Let $\mathcal{C} \in \Bbbk$-$\mathbf{Cat}$. We define a functor $Q_{\mathcal{C}}\colon \mathrm{Gr}(\Delta(\mathcal{C})) \to \mathcal{C}$ by

- $Q_{\mathcal{C}}({}_ix) := x$ for all ${}_ix \in \mathrm{Gr}(\Delta(\mathcal{C}))_0$; and
- $Q_{\mathcal{C}}((f_a)_{a\in I(i,j)}) := \sum_{a\in I(i,j)} f_a$ for all $(f_a)_{a\in I(i,j)} \in \mathrm{Gr}(\Delta(\mathcal{C}))({}_ix, {}_jy)$ and for all ${}_ix, {}_jy \in \mathrm{Gr}(\Delta(\mathcal{C}))_0$.

It is easy to verify that $Q_{\mathcal{C}}$ is a $\Bbbk$-functor.

Theorem 5.2. *The 2-functor* $\mathrm{Gr}\colon \overleftarrow{\mathrm{Colax}}(I, \Bbbk\text{-}\mathbf{Cat}) \to \Bbbk\text{-}\mathbf{Cat}$ *is a strict left 2-adjoint to the 2-functor* $\Delta\colon \Bbbk\text{-}\mathbf{Cat} \to \overleftarrow{\mathrm{Colax}}(I, \Bbbk\text{-}\mathbf{Cat})$*. The unit is given by the family of canonical morphisms* $(P_X, \phi_X)\colon X \to \Delta(\mathrm{Gr}(X))$ *indexed by* $X \in \overleftarrow{\mathrm{Colax}}(I, \Bbbk\text{-}\mathbf{Cat})$*, and the counit is given by the family of* $Q_{\mathcal{C}}\colon \mathrm{Gr}(\Delta(\mathcal{C})) \to \mathcal{C}$ *defined as above indexed by* $\mathcal{C} \in \Bbbk\text{-}\mathbf{Cat}$*.*

In particular, (P_X, ϕ_X) *has a strict universality in the comma category* $(X \downarrow \Delta)$*, i.e., for each* $(F, \psi)\colon X \to \Delta(\mathcal{C})$ *in* $\overleftarrow{\mathrm{Colax}}(I, \Bbbk\text{-}\mathbf{Cat})$ *with* $\mathcal{C} \in \Bbbk\text{-}\mathbf{Cat}$*, there exists a unique* $H\colon \mathrm{Gr}(X) \to \mathcal{C}$ *in* $\Bbbk\text{-}\mathbf{Cat}$ *such that the following is a commutative diagram in* $\overleftarrow{\mathrm{Colax}}(I, \Bbbk\text{-}\mathbf{Cat})$*:*

$$\begin{tikzcd} X \arrow[r, "{(F,\psi)}"] \arrow[d, "{(P_X,\phi_X)}"'] & \Delta(\mathcal{C}). \\ \Delta(\mathrm{Gr}(X)) \arrow[ur, dashed, "\Delta(H)"'] & \end{tikzcd}$$

Proof. For simplicity set $\eta := ((P_X, \phi_X))_{X \in \overleftarrow{\mathrm{Colax}}(I, \Bbbk\text{-}\mathbf{Cat})_0}$ and $\varepsilon := (Q_{\mathcal{C}})_{\mathcal{C} \in \Bbbk\text{-}\mathbf{Cat}_0}$.

Claim 1. $\Delta\varepsilon \cdot \eta\Delta = \mathbb{1}_\Delta$.

Indeed, Let $\mathcal{C} \in \Bbbk\text{-}\mathbf{Cat}$. It is enough to show that $\Delta(Q_{\mathcal{C}}) \cdot (P_{\Delta(\mathcal{C})}, \phi_{\Delta(\mathcal{C})}) = \mathbb{1}_{\Delta(\mathcal{C})}$. Now

$$\begin{aligned} \mathrm{LHS} &= \left((Q_{\mathcal{C}} P_{\Delta(\mathcal{C})}(i))_{i \in I_0}, (Q_{\mathcal{C}} \phi_{\Delta(\mathcal{C})}(a))_{a \in I_1}\right), \text{ and} \\ \mathrm{RHS} &= ((\mathbb{1}_{\mathcal{C}})_{i \in I_0}, (\mathbb{1}_{\mathbb{1}_{\mathcal{C}}})_{a \in I_1}). \end{aligned}$$

First entry: Let $i \in I$. Then $Q_{\mathcal{C}} P_{\Delta(\mathcal{C})}(i) = \mathbb{1}_{\mathcal{C}}$ because for each $x, y \in \mathcal{C}_0$ and each $f \in \mathcal{C}(x, y)$ we have $(Q_{\mathcal{C}} P_{\Delta(\mathcal{C})}(i))(x) = Q_{\mathcal{C}}({}_ix) = x$; and $(Q_{\mathcal{C}} P_{\Delta(\mathcal{C})}(i))(f) = (\delta_{a,\mathbf{1}_i} f \cdot ((\eta_{\Delta(\mathcal{C})})_i\, x))_{a \in I_1} = \sum_{a \in I(i,i)} \delta_{a,\mathbf{1}_i} f = f$.

Second entry: Let $a\colon i \to j$ in I. Then $Q_{\mathcal{C}} \phi_{\Delta(\mathcal{C})}(a) = \mathbb{1}_{\mathbb{1}_{\mathcal{C}}}$ because for each $x \in \mathcal{C}_0$ we have $Q_{\mathcal{C}}\left(\phi_{\Delta(\mathcal{C})}(a)x\right) = Q_{\mathcal{C}}\left((\delta_{b,a} \mathbb{1}_{\Delta(\mathcal{C})(a)x})_{b \in I(i,j)}\right) = \sum_{b \in I(i,j)} \delta_{b,a} \mathbb{1}_x = \mathbb{1}_x = \mathbb{1}_{\mathbb{1}_{\mathcal{C}} x}$. This shows that LHS = RHS.

Claim 2. $\varepsilon\,\mathrm{Gr} \cdot \mathrm{Gr}\,\eta = \mathbb{1}_{\mathrm{Gr}}$.

Indeed, let $X \in \overleftarrow{\mathrm{Colax}}(I, \Bbbk\text{-}\mathbf{Cat})$. It is enough to show that $Q_{\mathrm{Gr}(X)} \cdot \mathrm{Gr}(P_X, \phi_X) = \mathbb{1}_{\mathrm{Gr}(X)}$.

On objects: Let ${}_ix \in \mathrm{Gr}(X)_0$. Then $Q_{\mathrm{Gr}(X)}\left(\mathrm{Gr}(P_X, \phi_X)(x)\right) = Q_{\mathrm{Gr}(X)}({}_i(P_X(i)x)) = {}_ix$.

On morphisms: Let $f = (f_a)_{a \in I(i,j)}\colon {}_ix \to {}_jy$ be in $\mathrm{Gr}(X)$. Then $Q_{\mathrm{Gr}(X)}\, \mathrm{Gr}(P_X, \phi_X)(f) = Q_{\mathrm{Gr}(X)}((P_X(j)(f_a) \circ \phi_X(a)x)_{a \in I(i,j)}) = \sum_{a \in I(i,j)} P_X(j)(f_a) \circ \phi_X(a)x = (P_X, \phi_X)^{(1)}_{x,y}(f) = f$. Thus the claim holds.

The two claims above prove the assertion. □

Corollary 5.3. *Let* $(F, \psi)\colon X \to \Delta(\mathcal{C})$ *be in* $\overleftarrow{\mathrm{Colax}}(I, \Bbbk\text{-}\mathbf{Cat})$*. Then the following are equivalent.*

(1) (F, ψ) *is an* I*-covering;*

(2) *There exists an equivalence $H\colon \mathrm{Gr}(X) \to \mathcal{C}$ such that the diagram*

$$\begin{tikzcd} X \arrow[r, "{(F,\psi)}"] \arrow[d, "{(P_X,\phi_X)}"'] & \Delta(\mathcal{C}) \\ \Delta(\mathrm{Gr}(X)) \arrow[ur, "\Delta(H)"'] & \end{tikzcd}$$

is strictly commutative.

Proof. This immediately follows by Theorem 5.2 and Lemma 4.6. □

6. The Module colax functor

Let $X\colon I \to \Bbbk\text{-}\mathbf{Cat}$ be a colax functor. In this section we simplify the definition of the "module category" $\operatorname{Mod} X$ of X as a colax functor $I \to \Bbbk\text{-}\mathbf{Cat}$ given in our previous paper [6]. Recall that the *module category* $\operatorname{Mod}\mathcal{C}$ of a category $\mathcal{C} \in \Bbbk\text{-}\mathbf{Cat}$ is defined to be the functor category $\Bbbk\text{-}\mathbf{Cat}(\mathcal{C}^{\mathrm{op}}, \operatorname{Mod}\Bbbk)$, where $\operatorname{Mod}\Bbbk$ denotes the category of $\Bbbk$-modules. Since $\Bbbk\text{-}\mathbf{Cat}$ is a 2-category, this is extended to a representable 2-functor

$$\operatorname{Mod}' := \Bbbk\text{-}\mathbf{Cat}((\text{-})^{\mathrm{op}}, \operatorname{Mod}\Bbbk)\colon \Bbbk\text{-}\mathbf{Cat} \to \Bbbk\text{-}\mathbf{Ab}^{\mathrm{coop}}$$

(see Notation 2.6). As is easily seen the composite $\operatorname{Mod}' \circ X$ turns out to be a colax functor $I \to \Bbbk\text{-}\mathbf{Ab}^{\mathrm{coop}}$, i.e., a contravariant lax functor $I \to \Bbbk\text{-}\mathbf{Ab}$. When X is a group action, namely when I is a group G and $X\colon G \to \Bbbk\text{-}\mathbf{Cat}$ is a functor, the usual module category $\operatorname{Mod} X$ with a G-action of X was defined to be the composite functor $\operatorname{Mod} X := \operatorname{Mod}' \circ X \circ i$, where $i\colon G \to G$ is the group anti-isomorphism defined by $x \mapsto x^{-1}$ for all $x \in G$. In this way we can change $\operatorname{Mod}' \circ X$ to a covariant one. But in general we cannot assume the existence of such an isomorphism i. Instead in this paper we will use a covariant "pseudofunctor" $\operatorname{Mod}\colon \Bbbk\text{-}\mathbf{Cat} \to \Bbbk\text{-}\mathbf{Ab}$ defined below and will define $\operatorname{Mod} X$ as the composite $\operatorname{Mod} \circ X$, which can be seen as a "lax" extended version of the module category construction of a category with a G-action stated above. We start with a notion of colax functors from a 2-category to a 2-category. Compare our definitions of colax functors, left transformations (1-morphisms) and 2-morphisms in the setting of 2-categories given below with definitions of morphisms, transformations and modifications in the setting of bicategories (see Leinster [13] for instance).

Definition 6.1. Let $\mathbf{B}$ and $\mathbf{C}$ be 2-categories.

(1) A *colax functor* from $\mathbf{B}$ to $\mathbf{C}$ is a triple (X, η, θ) of data:

- a triple $X = (X_0, X_1, X_2)$ of maps $X_i\colon \mathbf{B}_i \to \mathbf{C}_i$ ($\mathbf{B}_i$ denotes the collection of i-morphisms of $\mathbf{B}$ for each $i = 0, 1, 2$) preserving domains and codomains of all 1-morphisms and 2-morphisms (i.e. $X_1(\mathbf{B}_1(i,j)) \subseteq \mathbf{C}_1(X_0 i, X_0 j)$ for all $i, j \in \mathbf{B}_0$ and $X_2(\mathbf{B}_2(a,b)) \subseteq \mathbf{C}_2(X_1 a, X_1 b)$ for all $a, b \in \mathbf{B}_1$ (we omit the subscripts of X below));
- a family $\eta := (\eta_i)_{i\in\mathbf{B}_0}$ of 2-morphisms $\eta_i\colon X(\mathbb{1}_i) \Rightarrow \mathbb{1}_{X(i)}$ in $\mathbf{C}$ indexed by $i \in \mathbf{B}_0$; and
- a family $\theta := (\theta_{b,a})_{(b,a)}$ of 2-morphisms $\theta_{b,a}\colon X(ba) \Rightarrow X(b)X(a)$ in $\mathbf{C}$ indexed by $(b,a) \in \operatorname{com}(\mathbf{B}) := \{(b,a) \in \mathbf{B}_1 \times \mathbf{B}_1 \mid ba \text{ is defined}\}$

satisfying the axioms:

(i) $(X_1, X_2)\colon \mathbf{B}(i,j) \to \mathbf{C}(X_0 i, X_0 j)$ is a functor for all $i, j \in \mathbf{B}_0$;

(ii) For each $a\colon i \to j$ in $\mathbf{B}_1$ the following are commutative:

$$\begin{tikzcd} X(a\mathbb{1}_i) \arrow[r, Rightarrow, "\theta_{a,\mathbb{1}_i}"] \arrow[dr, equal] & X(a)X(\mathbb{1}_i) \arrow[d, Rightarrow, "X(a)\eta_i"] \\ & X(a)\mathbb{1}_{X(i)} \end{tikzcd} \quad \text{and} \quad \begin{tikzcd} X(\mathbb{1}_j a) \arrow[r, Rightarrow, "\theta_{\mathbb{1}_j,a}"] \arrow[dr, equal] & X(\mathbb{1}_j)X(a) \arrow[d, Rightarrow, "\eta_j X(a)"] \\ & \mathbb{1}_{X(j)}X(a) \end{tikzcd} ;$$

(iii) For each $i \xrightarrow{a} j \xrightarrow{b} k \xrightarrow{c} l$ in $\mathbf{B}_1$ the following is commutative:

$$\begin{tikzcd} X(cba) \arrow[r, Rightarrow, "\theta_{c,ba}"] \arrow[d, Rightarrow, "\theta_{cb,a}"'] & X(c)X(ba) \arrow[d, Rightarrow, "X(c)\theta_{b,a}"] \\ X(cb)X(a) \arrow[r, Rightarrow, "\theta_{c,b}X(a)"'] & X(c)X(b)X(a) \end{tikzcd} \quad ; \text{ and}$$

(iv) For each $a, a'\colon i \to j$ and $b, b'\colon j \to k$ in $\mathbf{B}_1$ and each $\alpha\colon a \to a'$, $\beta\colon b \to b'$ in $\mathbf{B}_2$ the following is commutative:

$$\begin{tikzcd} X(ba) \arrow[r, Rightarrow, "\theta_{b,a}"] \arrow[d, Rightarrow, "X(\beta*\alpha)"'] & X(b)X(a) \arrow[d, Rightarrow, "X(\beta)*X(\alpha)"] \\ X(b'a') \arrow[r, Rightarrow, "\theta_{b',a'}"] & X(b')X(a'). \end{tikzcd}$$

(2) A *lax functor* from $\mathbf{B}$ to $\mathbf{C}$ is a colax functor from $\mathbf{B}$ to $\mathbf{C}^{\mathrm{co}}$ (see Notation 2.6).

(3) A *pseudofunctor* from $\mathbf{B}$ to $\mathbf{C}$ is a colax functor with all η_i and $\theta_{b,a}$ 2-isomorphisms.

(4) We define a 2-category $\overleftarrow{\mathrm{Colax}}(\mathbf{B}, \mathbf{C})$ having all the colax functors $\mathbf{B} \to \mathbf{C}$ as the objects as follows.

1-morphisms. Let $X = (X, \eta, \theta)$, $X' = (X', \eta', \theta')$ be colax functors from $\mathbf{B}$ to $\mathbf{C}$. A 1-*morphism* (called a *left transformation*) from X to X' is a pair (F, ψ) of data

- a family $F := (F(i))_{i\in\mathbf{B}_0}$ of 1-morphisms $F(i)\colon X(i) \to X'(i)$ in $\mathbf{C}$; and
- a family $\psi := (\psi(a))_{a\in\mathbf{B}_1}$ of 2-morphisms $\psi(a)\colon X'(a)F(i) \Rightarrow F(j)X(a)$

$$\begin{tikzcd} X(i) \arrow[r, "F(i)"] \arrow[d, "X(a)"'] & X'(i) \arrow[d, "X'(a)"] \arrow[dl, Rightarrow, "\psi(a)"'] \\ X(j) \arrow[r, "F(j)"'] & X'(j) \end{tikzcd}$$

in $\mathbf{C}$ indexed by $a\colon i \to j$ in $\mathbf{B}_1$ with the property that

(0) for each $\alpha\colon a \Rightarrow b$ in $\mathbf{B}(i,j)$ the following is commutative:

$$\begin{tikzcd} X'(a)F(i) \arrow[r, Rightarrow, "X'(\alpha)F(i)"] \arrow[d, Rightarrow, "\psi(a)"'] & X'(b)F(i) \arrow[d, Rightarrow, "\psi(b)"] \\ F(j)X(a) \arrow[r, Rightarrow, "F(j)X(\alpha)"'] & F(j)X(b), \end{tikzcd} \tag{6.1}$$

thus a family of natural transformations of functors

$$\begin{tikzcd} \mathbf{B}(i,j) \arrow[r, "X'"] \arrow[d, "X"'] & \mathbf{C}(X'(i), X'(j)) \arrow[d, "\mathbf{C}(F(i),X'(j))"] \arrow[dl, Rightarrow, "\psi_{ij}"'] \\ \mathbf{C}(X(i), X(j)) \arrow[r, "\mathbf{C}(X(i),F(j))"'] & \mathbf{C}(X(i), X'(j)) \end{tikzcd} \quad (i, j \in \mathbf{B}_0)$$

satisfying the axioms

(a) For each $i \in \mathbf{B}_0$ the following is commutative:

$$\begin{tikzcd} X'(\mathbb{1}_i)F(i) \arrow[r, Rightarrow, "\psi(\mathbf{1}_i)"] \arrow[d, Rightarrow, "\eta'_i F(i)"'] & F(i)X(\mathbb{1}_i) \arrow[d, Rightarrow, "F(i)\eta_i"] \\ \mathbb{1}_{X'(i)}F(i) \arrow[r, equal] & F(i)\mathbb{1}_{X(i)} \end{tikzcd} \quad ; \text{ and}$$

(b) For each $i \xrightarrow{a} j \xrightarrow{b} k$ in $\mathbf{B}_1$ the following is commutative:

$$\begin{tikzcd} X'(ba)F(i) \arrow[r, Rightarrow, "\theta'_{b,a}F(i)"] \arrow[d, Rightarrow, "\psi(ba)"'] & X'(b)X'(a)F(i) \arrow[r, Rightarrow, "X'(b)\psi(a)"] & X'(b)F(j)X(a) \arrow[d, Rightarrow, "\psi(b)X(a)"] \\ F(k)X(ba) \arrow[rr, Rightarrow, "F(k)\,\theta_{b,a}"'] & & F(k)X(b)X(a). \end{tikzcd}$$

2-morphisms. Let $X = (X, \eta, \theta)$, $X' = (X', \eta', \theta')$ be colax functors from $\mathbf{B}$ to $\mathbf{C}$, and (F, ψ), (F', ψ') 1-morphisms from X to X'. A 2-*morphism* from (F, ψ) to (F', ψ') is a family $\zeta = (\zeta(i))_{i\in\mathbf{B}_0}$ of 2-morphisms $\zeta(i)\colon F(i) \Rightarrow F'(i)$ in $\mathbf{C}$ indexed by $i \in \mathbf{B}_0$ such that the following is commutative for all $a\colon i \to j$ in $\mathbf{B}_1$:

$$\begin{tikzcd} X'(a)F(i) \arrow[r, Rightarrow, "X'(a)\zeta(i)"] \arrow[d, Rightarrow, "\psi(a)"'] & X'(a)F'(i) \arrow[d, Rightarrow, "\psi'(a)"] \\ F(j)X(a) \arrow[r, Rightarrow, "\zeta(j)X(a)"] & F'(j)X(a). \end{tikzcd}$$

Composition of 1-morphisms. Let $X = (X, \eta, \theta)$, $X' = (X', \eta', \theta')$ and $X'' = (X'', \eta'', \theta'')$ be colax functors from $\mathbf{B}$ to $\mathbf{C}$, and let $(F, \psi)\colon X \to X'$, $(F', \psi')\colon X' \to X''$ be 1-morphisms. Then the composite $(F', \psi')(F, \psi)$ of (F, ψ) and (F', ψ') is a 1-morphism from X to X'' defined by

$$(F', \psi')(F, \psi) := (F'F, \psi' \circ \psi),$$

where $F'F := ((F'(i)F(i))_{i\in\mathbf{B}_0}$ and for each $a\colon i \to j$ in $\mathbf{B}$, $(\psi' \circ \psi)(a) := F'(j)\psi(a) \circ \psi'(a)F(i)$ is the pasting of the diagram

$$\begin{tikzcd} X(i) \arrow[r, "F(i)"] \arrow[d, "X(a)"'] & X'(i) \arrow[r, "F'(i)"] \arrow[d, "X'(a)"] \arrow[dl, Rightarrow, "\psi(a)"'] & X''(i) \arrow[d, "X''(a)"] \arrow[dl, Rightarrow, "\psi'(a)"'] \\ X(j) \arrow[r, "F(j)"'] & X'(j) \arrow[r, "F'(j)"'] & X''(j). \end{tikzcd}$$

Remark 6.2. (1) Note that a (strict) 2-functor from $\mathbf{B}$ to $\mathbf{C}$ is a pseudofunctor with all η_i and $\theta_{b,a}$ identities.

(2) By regarding the category I as a 2-category with all 2-morphisms identities, the definition (1) of colax functors above coincides with Definition 2.1.

(3) When $\mathbf{B} = I$, the definition (4) of $\overleftarrow{\mathrm{Colax}}(\mathbf{B}, \mathbf{C})$ above coincides with that of $\overleftarrow{\mathrm{Colax}}(I, \mathbf{C})$ given before.

Example 6.3. (1) Since $\Bbbk$-$\mathbf{Cat}$ is a 2-category, $\mathrm{Mod}' := \Bbbk\text{-}\mathbf{Cat}((\text{-})^{\mathrm{op}}, \mathrm{Mod}\,\Bbbk) \colon \Bbbk\text{-}\mathbf{Cat} \to \Bbbk\text{-}\mathbf{Ab}^{\mathrm{coop}}$ is a 2-functor, which we can regard as a contravariant lax functor

$$\mathrm{Mod}' := \Bbbk\text{-}\mathbf{Cat}((\text{-})^{\mathrm{op}}, \mathrm{Mod}\,\Bbbk) \colon \Bbbk\text{-}\mathbf{Cat} \to \Bbbk\text{-}\mathbf{Ab}.$$

(2) We define a pseudofunctor $\mathrm{Mod} \colon \Bbbk\text{-}\mathbf{Cat} \to \Bbbk\text{-}\mathbf{Ab}$ as follows.

- For each $\mathcal{C} \in \Bbbk\text{-}\mathbf{Cat}_0$ we set $\mathrm{Mod}\,\mathcal{C} := \mathrm{Mod}'\,\mathcal{C}$.
- For each $F \colon \mathcal{C} \to \mathcal{C}'$ in $\Bbbk\text{-}\mathbf{Cat}_1$ we set $\mathrm{Mod}\,F := \text{-} \otimes_{\mathcal{C}} \overline{F} \colon \mathrm{Mod}\,\mathcal{C} \to \mathrm{Mod}\,\mathcal{C}'$, where $\overline{F}$ is the $\mathcal{C}$-$\mathcal{C}'$-bimodule defined by $\overline{F}(x, y) := \mathcal{C}'(y, F(x))$ for all $x \in \mathcal{C}_0$, $y \in \mathcal{C}'_0$, which we sometimes write as $\overline{F} := \mathcal{C}'(?, F(\text{-}))$.
- For each $\alpha \colon F \Rightarrow G$ in $\Bbbk\text{-}\mathbf{Cat}_2$ (with $F, G \colon \mathcal{C} \to \mathcal{C}'$ in $\Bbbk\text{-}\mathbf{Cat}_1$) we define $\mathrm{Mod}\,\alpha \colon \mathrm{Mod}\,F \Rightarrow \mathrm{Mod}\,G$ by setting $(\mathrm{Mod}\,\alpha)x := \mathcal{C}'(?, \alpha x) \colon \mathcal{C}'(?, Fx) \Rightarrow \mathcal{C}'(?, Gx)$ for all $x \in \mathcal{C}_0$.
- For each $\mathcal{C} \in \Bbbk\text{-}\mathbf{Cat}$ we define $\eta_{\mathcal{C}} \colon \mathrm{Mod}\,\mathbb{1}_{\mathcal{C}} \Rightarrow \mathbb{1}_{\mathrm{Mod}\,\mathcal{C}}$ by setting $\eta_{\mathcal{C}} M \colon M \otimes_{\mathcal{C}} \mathcal{C}(?, \text{-}) \to M$ to be the canonical isomorphisms for all $M \in \mathrm{Mod}\,\mathcal{C}$.
- For each pair of functors $\mathcal{C} \xrightarrow{F} \mathcal{C}' \xrightarrow{G} \mathcal{C}''$ in $\Bbbk\text{-}\mathbf{Cat}$ we define $\theta_{G,F} \colon \mathrm{Mod}\,GF \Rightarrow \mathrm{Mod}\,G \circ \mathrm{Mod}\,F$ as the inverse of the canonical isomorphism

$$\text{-} \otimes_{\mathcal{C}} \mathcal{C}'(?, F(\text{-})) \otimes_{\mathcal{C}'} \mathcal{C}''(?, G(\text{-})) \Rightarrow \text{-} \otimes_{\mathcal{C}} \mathcal{C}''(?, GF(\text{-})).$$

It is straightforward to check that this defines a pseudofunctor.

(3) Denote by $\Bbbk$-$\mathbf{ModCat}$ the 2-subcategory of $\Bbbk$-$\mathbf{Ab}$ consisting of the following:

- objects: $\mathrm{Mod}\,\mathcal{C}$ with $\mathcal{C} \in \Bbbk\text{-}\mathbf{Cat}_0$,
- 1-morphisms: functors between objects having exact right adjoints, and
- 2-morphisms: all natural transformations between 1-morphisms.

Then note that the pseudofunctor $\mathrm{Mod} \colon \Bbbk\text{-}\mathbf{Cat} \to \Bbbk\text{-}\mathbf{Ab}$ defined above can be seen as a pseudofunctor $\Bbbk\text{-}\mathbf{Cat} \to \Bbbk\text{-}\mathbf{ModCat}$. For each $\mathrm{Mod}\,\mathcal{C}$ with $\mathcal{C} \in \Bbbk\text{-}\mathbf{Cat}_0$ we denote by $\mathcal{K}_p(\mathrm{Mod}\,\mathcal{C})$ the full subcategory of the homotopy category $\mathcal{K}(\mathrm{Mod}\,\mathcal{C})$ of $\mathrm{Mod}\,\mathcal{C}$ consisting of *homotopically projective* objects M, i.e., objects M such that $\mathcal{K}(\mathrm{Mod}\,\mathcal{C})(M, A) = 0$ for all acyclic objects A. Recall that there is a natural embedding $\mathbf{j}_{\mathcal{C}} \colon \mathcal{K}_p(\mathrm{Mod}\,\mathcal{C}) \to \mathcal{D}(\mathrm{Mod}\,\mathcal{C})$ having a left adjoint $\mathbf{p}_{\mathcal{C}}$ such that there exists a quasi-isomorphism $\eta_{\mathcal{C}} M \colon \mathbf{j}_{\mathcal{C}} \mathbf{p}_{\mathcal{C}} M \to M$ for each $M \in \mathcal{D}(\mathrm{Mod}\,\mathcal{C})$ and that $\mathbf{p}_{\mathcal{C}} \mathbf{j}_{\mathcal{C}} = \mathbb{1}_{\mathcal{K}_p(\mathrm{Mod}\,\mathcal{C})}$. Then we can define a pseudofunctor $\mathcal{D} \colon \Bbbk\text{-}\mathbf{ModCat} \to \Bbbk\text{-}\mathbf{Tri}$ as follows.

- For each $\mathrm{Mod}\,\mathcal{C}$ in $\Bbbk\text{-}\mathbf{ModCat}_0$ with $\mathcal{C} \in \Bbbk\text{-}\mathbf{Cat}$ we set $\mathcal{D}(\mathrm{Mod}\,\mathcal{C})$ to be the derived category of $\mathrm{Mod}\,\mathcal{C}$.
- For each $F \colon \mathrm{Mod}\,\mathcal{C} \to \mathrm{Mod}\,\mathcal{C}'$ in $\Bbbk\text{-}\mathbf{ModCat}_1$, F naturally induces a functor $\mathcal{K}F \colon \mathcal{K}(\mathrm{Mod}\,\mathcal{C}) \to \mathcal{K}(\mathrm{Mod}\,\mathcal{C}')$, which restricts to a functor $\mathcal{K}_p F \colon \mathcal{K}_p(\mathrm{Mod}\,\mathcal{C}) \to \mathcal{K}_p(\mathrm{Mod}\,\mathcal{C}')$ because F has an exact right adjoint. Then we set $\mathcal{D}F$ to be the

left derived functor $\mathbf{L}F\colon \mathcal{D}(\operatorname{Mod}\mathcal{C}) \to \mathcal{D}(\operatorname{Mod}\mathcal{C}')$ of F, which is defined as the composite $\mathbf{L}F := \mathbf{j}_{\mathcal{C}'}(\mathcal{K}_p F)\mathbf{p}_{\mathcal{C}}$.

- For each $\alpha\colon F \Rightarrow F'$ in $\Bbbk$-$\mathbf{ModCat}_2$ with $F, F'\colon \operatorname{Mod}\mathcal{C} \to \operatorname{Mod}\mathcal{C}'$ in $\Bbbk$-$\mathbf{ModCat}_1$, α naturally induces a natural transformation $\mathcal{K}_p\alpha\colon \mathcal{K}_pF \Rightarrow \mathcal{K}_pF'$. Then we define $\mathcal{D}\alpha := \mathbf{j}_{\mathcal{C}'}(\mathcal{K}_p\alpha)\mathbf{p}_{\mathcal{C}}$.
- We define $\eta_{\operatorname{Mod}\mathcal{C}}\colon \mathcal{D}(\mathbb{1}_{\operatorname{Mod}\mathcal{C}})(= \mathbf{j}_{\mathcal{C}}\mathbf{p}_{\mathcal{C}}) \Rightarrow \mathbb{1}_{\mathcal{D}(\operatorname{Mod}\mathcal{C})}$ by $\eta_{\operatorname{Mod}\mathcal{C}} := (\eta_{\mathcal{C}}M)_{M\in\mathcal{D}(\operatorname{Mod}\mathcal{C})}$.
- Note that for each $\operatorname{Mod}\mathcal{C} \xrightarrow{F} \operatorname{Mod}\mathcal{C}' \xrightarrow{F'} \operatorname{Mod}\mathcal{C}''$ in $\Bbbk$-$\mathbf{ModCat}_1$ we have $\mathbf{L}(F' \circ F) = \mathbf{L}F' \circ \mathbf{L}F$ because $\mathbf{p}_{\mathcal{C}'}\mathbf{j}_{\mathcal{C}'} = \mathbb{1}_{\mathcal{K}_p(\operatorname{Mod}\mathcal{C})}$. We define $\theta_{F',F}\colon \mathbf{L}(F' \circ F) \Rightarrow \mathbf{L}F' \circ \mathbf{L}F$ as the identity $\mathbb{1}_{\mathbf{L}(F'\circ F)}$.

It is straightforward to check that this defines a pseudofunctor.

Example 6.4. (1) We define a pseudofunctor $\operatorname{prj}\colon \Bbbk\text{-}\mathbf{Cat} \to \Bbbk\text{-}\mathbf{add}$ as the subpseudofunctor of $\operatorname{Mod}\colon \Bbbk\text{-}\mathbf{Cat} \to \Bbbk\text{-}\mathbf{Ab} \hookrightarrow \Bbbk\text{-}\mathbf{add}$ by setting $\operatorname{prj}\mathcal{C}$ to be the full subcategory of $\operatorname{Mod}\mathcal{C}$ consisting of finitely generated projective $\mathcal{C}$-modules for all $\mathcal{C} \in \Bbbk\text{-}\mathbf{Cat}_0$, where $\Bbbk$-$\mathbf{add}$ is the full 2-subcategory of $\Bbbk$-$\mathbf{Cat}$ consisting of additive $\Bbbk$-categories. Then for each $F\colon \mathcal{C} \to \mathcal{C}'$ in $\Bbbk$-$\mathbf{Cat}_1$ and each $x \in \mathcal{C}_0$ we have

$$(\operatorname{prj} F)(\mathcal{C}(\text{-}, x)) = \mathcal{C}(\text{-}, x) \otimes_{\mathcal{C}} \overline{F} \cong \mathcal{C}'(\text{-}, F(x)). \tag{6.2}$$

Note that we can define two 2-functors $\oplus\colon \Bbbk\text{-}\mathbf{Cat} \to \Bbbk\text{-}\mathbf{add}$ and $\operatorname{sic}\colon \Bbbk\text{-}\mathbf{add} \to \Bbbk\text{-}\mathbf{add}$ by forming formal additive hulls (see e.g., [2, Subsection 4.1]) and by taking split idempotent completions (see e.g., [4, Definition 3.1]), respectively. Then the Yoneda embeddings $Y_{\mathcal{C}}\colon \mathcal{C} \to \operatorname{prj}\mathcal{C}$, $x \mapsto \mathcal{C}(\text{-}, x)$ ($\mathcal{C} \in \Bbbk\text{-}\mathbf{Cat}_0$) induce a natural 2-isomorphism $Y\colon \operatorname{sic}\circ\oplus \Rightarrow \operatorname{prj}$:

$$\begin{tikzcd} \Bbbk\text{-}\mathbf{Cat} \arrow[rr, "\operatorname{prj}"] \arrow[dr, "\oplus"'] & \arrow[d, phantom, "\cong\Uparrow Y"] & \Bbbk\text{-}\mathbf{add}. \\ & \Bbbk\text{-}\mathbf{add} \arrow[ur, "\operatorname{sic}"'] & \end{tikzcd}$$

(2) A 2-functor $\mathcal{K}^{\mathrm{b}}\colon \Bbbk\text{-}\mathbf{add} \to \Bbbk\text{-}\mathbf{Tri}$ is canonically defined by setting $\mathcal{K}^{\mathrm{b}}(\mathcal{C})$ to be the homotopy category of bounded complexes in $\mathcal{C}$ for all $\mathcal{C} \in \Bbbk\text{-}\mathbf{add}$. Then the composite pseudofunctor $\mathcal{K}^{\mathrm{b}} \circ \operatorname{prj}\colon \Bbbk\text{-}\mathbf{Cat} \to \Bbbk\text{-}\mathbf{Tri}$ turns out to be a subpseudofunctor of $\mathcal{D} \circ \operatorname{Mod}\colon \Bbbk\text{-}\mathbf{Cat} \to \Bbbk\text{-}\mathbf{Tri}$.

The following is a useful tool to define new colax functors from an old one by composing with pseudofunctors. The proof will be given in the last section.

Theorem 6.5. *Let* $\mathbf{B}, \mathbf{C}$ *and* $\mathbf{D}$ *be* 2*-categories and* $V\colon \mathbf{C} \to \mathbf{D}$ *a pseudofunctor. Then the obvious correspondence* (*see subsection 9.1 for details*)

$$\overleftarrow{\operatorname{Colax}}(\mathbf{B}, V)\colon \overleftarrow{\operatorname{Colax}}(\mathbf{B}, \mathbf{C}) \to \overleftarrow{\operatorname{Colax}}(\mathbf{B}, \mathbf{D})$$

turns out to be a pseudofunctor.

Definition 6.6. Let $X = (X, \eta, \theta) \in \overleftarrow{\operatorname{Colax}}(I, \Bbbk\text{-}\mathbf{Cat})$.

(1) We define the *module colax functor* $\operatorname{Mod}X = (\operatorname{Mod}X, \operatorname{Mod}\eta, \operatorname{Mod}\theta)\colon I \to \Bbbk\text{-}\mathbf{ModCat}$ of X as the composite $\operatorname{Mod}X := \operatorname{Mod}\circ X = \overleftarrow{\operatorname{Colax}}(I, \operatorname{Mod})(X)\colon I \xrightarrow{X} \Bbbk\text{-}\mathbf{Cat} \xrightarrow{\operatorname{Mod}} \Bbbk\text{-}\mathbf{ModCat}$. By applying Theorem 6.5 to $\mathbf{B} := I$, $\mathbf{C} := \Bbbk\text{-}\mathbf{Cat}$, $\mathbf{D} :=$

$\Bbbk$-**ModCat** and $V :=$ Mod (Example 6.3(2)) we see that $\operatorname{Mod} X \in \overleftarrow{\mathrm{Colax}}(I, \Bbbk\text{-}\mathbf{ModCat})$. Then we have

- for each $i \in I_0$, $(\operatorname{Mod} X)(i) = \operatorname{Mod}(X(i))$; and
- for each $a \colon i \to j$ in I the functor $(\operatorname{Mod} X)(a) \colon (\operatorname{Mod} X)(i) \to (\operatorname{Mod} X)(j)$ is given by $(\operatorname{Mod} X)(a) = \text{-} \otimes_{X(i)} \overline{X(a)}$, where $\overline{X(a)}$ is an $X(i)$-$X(j)$-bimodule defined by
$$\overline{X(a)}(x, y) := X(j)(y, X(a)(x))$$
for all $x \in X(i)_0$ and $y \in X(j)_0$.

(2) By Theorem 6.5 and Example 6.3 we can define a colax functor $\mathcal{D}(\operatorname{Mod} X) \in \overleftarrow{\mathrm{Colax}}(I, \Bbbk\text{-}\mathbf{Tri})$ as the composite $\mathcal{D}(\operatorname{Mod} X) := \mathcal{D} \circ \operatorname{Mod} X$, which we call the *derived module colax functor* of X. Then for each $a \colon i \to j$ in I, $\mathcal{D}(\operatorname{Mod} X)(i) \xrightarrow{\mathcal{D}(\operatorname{Mod} X)(a)} \mathcal{D}(\operatorname{Mod} X)(j)$ is equal to
$$\mathcal{D}(\operatorname{Mod} X(i)) \xrightarrow{\text{-} \overset{\mathbf{L}}{\otimes}_{X(i)} \overline{X(a)}} \mathcal{D}(\operatorname{Mod} X(j)).$$

(3) By Theorem 6.5 and Example 6.4 we can define a pseudofunctor
$$\overleftarrow{\mathrm{Colax}}(I, \mathcal{K}^{\mathrm{b}} \circ \operatorname{prj}) \colon \overleftarrow{\mathrm{Colax}}(I, \Bbbk\text{-}\mathbf{Cat}) \to \overleftarrow{\mathrm{Colax}}(I, \Bbbk\text{-}\mathbf{Tri})$$
sending each $X \in \overleftarrow{\mathrm{Colax}}(I, \Bbbk\text{-}\mathbf{Cat})$ to $\mathcal{K}^{\mathrm{b}}(\operatorname{prj} X)$. By the remark in Example 6.4(2) $\mathcal{K}^{\mathrm{b}}(\operatorname{prj} X)$ is a colax subfunctor of $\mathcal{D}(\operatorname{Mod} X)$.

Remark 6.7. Let $\mathcal{C} \in \Bbbk\text{-}\mathbf{Cat}_0$. Then it is obvious by definitions that
$$\Delta(\mathcal{K}^{\mathrm{b}}(\operatorname{prj} \mathcal{C})) = \mathcal{K}^{\mathrm{b}}(\operatorname{prj} \Delta(\mathcal{C})).$$

Proposition 6.8. *The pseudofunctor $\mathcal{K}^{\mathrm{b}} \circ \operatorname{prj}$ preserves I-precoverings, that is, if $(F, \psi) \colon X \to \Delta(\mathcal{C})$ is an I-precovering in $\overleftarrow{\mathrm{Colax}}(I, \Bbbk\text{-}\mathbf{Cat})$ with $\mathcal{C} \in \Bbbk\text{-}\mathbf{Cat}_0$, then so is $\mathcal{K}^{\mathrm{b}}(\operatorname{prj}(F, \psi)) \colon \mathcal{K}^{\mathrm{b}}(\operatorname{prj} X) \to \Delta(\mathcal{K}^{\mathrm{b}}(\operatorname{prj} \mathcal{C}))$ in $\overleftarrow{\mathrm{Colax}}(I, \Bbbk\text{-}\mathbf{Tri})$.*

Proof. It is straightforward to verify that the 2-functors $\oplus$, sic and $\mathcal{K}^{\mathrm{b}}$ defined in Example 6.4 preserve I-precoverings. Then the assertion follows from the natural 2-isomorphism $Y \colon \operatorname{sic} \circ \oplus \Rightarrow \operatorname{prj}$. $\square$

7. Derived equivalences of colax functors

In this section we recall necessary terminologies and the main theorem in our previous paper [6]. First we cite the following. See [6] for the proof.

Lemma 7.1. *Let $\mathbf{C}$ be a 2-category and $(F, \psi) \colon X \to X'$ a 1-morphism in the 2-category $\overleftarrow{\mathrm{Colax}}(I, \mathbf{C})$. Then (F, ψ) is an equivalence in $\overleftarrow{\mathrm{Colax}}(I, \mathbf{C})$ if and only if*

1. *For each $i \in I_0$, $F(i)$ is an equivalence in $\mathbf{C}$; and*
2. *For each $a \in I_1$, $\psi(a)$ is a 2-isomorphism in $\mathbf{C}$ (namely, (F, ψ) is I-equivariant).*

Definition 7.2. Let $X, X' \in \overleftarrow{\mathrm{Colax}}(I, \Bbbk\text{-}\mathbf{Cat})$. Then X and X' are said to be *derived equivalent* if $\mathcal{D}(\operatorname{Mod} X)$ and $\mathcal{D}(\operatorname{Mod} X')$ are equivalent in the 2-category $\overleftarrow{\mathrm{Colax}}(I, \Bbbk\text{-}\mathbf{Tri})$.

By Lemma 7.1 we obtain the following.

Proposition 7.3. *Let $X, X' \in \overleftarrow{\mathrm{Colax}}(I, \Bbbk\text{-}\mathbf{Cat})$. Then X and X' are derived equivalent if and only if there exists a 1-morphism $(F, \psi)\colon \mathcal{D}(\mathrm{Mod}\, X) \to \mathcal{D}(\mathrm{Mod}\, X')$ in $\overleftarrow{\mathrm{Colax}}(I, \Bbbk\text{-}\mathbf{Tri})$ such that*

1. *For each $i \in I_0$, $F(i)$ is a triangle equivalence; and*
2. *For each $a \in I_1$, $\psi(a)$ is a natural isomorphism (i.e., (F, ψ) is I-equivariant).*

A $\Bbbk$-category $\mathcal{A}$ is called $\Bbbk$-*projective* (resp. $\Bbbk$-*flat*) if $\mathcal{A}(x, y)$ are projective (resp. flat) $\Bbbk$-modules for all $x, y \in \mathcal{A}_0$.

Definition 7.4. Let $X\colon I \to \Bbbk\text{-}\mathbf{Cat}$ be a colax functor.

1. X is called $\Bbbk$-*projective* (resp. $\Bbbk$-*flat*) if $X(i)$ are $\Bbbk$-projective (resp. $\Bbbk$-flat) for all $i \in I_0$.
2. A colax subfunctor $\mathcal{T}$ of of $\mathcal{K}^{\mathrm{b}}(\mathrm{prj}\, X)$ is called *tilting* if for each $i \in I_0$, $\mathcal{T}(i)$ is a tilting subcategory of $\mathcal{K}^{\mathrm{b}}(\mathrm{prj}\, X(i))$, namely,
   - $\mathcal{K}^{\mathrm{b}}(\mathrm{prj}\, X(i))(U, V[n]) = 0$ for all $U, V \in \mathcal{T}(i)_0$ and $0 \neq n \in \mathbb{Z}$; and
   - the smallest thick subcategory of $\mathcal{K}^{\mathrm{b}}(\mathrm{prj}\, X(i))$ containing $\mathcal{T}(i)$ is equal to $\mathcal{K}^{\mathrm{b}}(\mathrm{prj}\, X(i))$.
3. A tilting colax subfunctor $\mathcal{T}$ of $\mathcal{K}^{\mathrm{b}}(\mathrm{prj}\, X)$ with an I-equivariant inclusion $(\sigma, \rho)\colon \mathcal{T} \hookrightarrow \mathcal{K}^{\mathrm{b}}(\mathrm{prj}\, X)$ is called a *tilting colax functor* for X.

The following was our main theorem in [6] that gives a generalization of the Morita type theorem characterizing derived equivalences of categories by Rickard [14] and Keller [11] in our setting.

Theorem 7.5. *Let $X, X' \in \overleftarrow{\mathrm{Colax}}(I, \Bbbk\text{-}\mathbf{Cat})$. Consider the following conditions.*

1. *X and X' are derived equivalent.*
2. *$\mathcal{K}^{\mathrm{b}}(\mathrm{prj}\, X)$ and $\mathcal{K}^{\mathrm{b}}(\mathrm{prj}\, X')$ are equivalent in $\overleftarrow{\mathrm{Colax}}(I, \Bbbk\text{-}\mathbf{Tri})$.*
3. *There exists a tilting colax functor $\mathcal{T}$ for X such that $\mathcal{T}$ and X' are equivalent in $\overleftarrow{\mathrm{Colax}}(I, \Bbbk\text{-}\mathbf{Cat})$.*

Then

(a) (1) *implies* (2).
(b) (2) *implies* (3).
(c) *If X' is $\Bbbk$-projective, then* (3) *implies* (1).

8. Derived equivalences of Grothendieck constructions

First we cite the statement [11, Corollary 9.2] in the $\Bbbk$-category case.

Theorem 8.1 (Keller). *Let $\mathcal{A}$ and $\mathcal{B}$ be $\Bbbk$-categories and assume that $\mathcal{A}$ is $\Bbbk$-flat. Then the following are equivalent.*

1. *$\mathcal{A}$ and $\mathcal{B}$ are derived equivalent.*
2. *$\mathcal{B}$ is equivalent to a tilting subscategory of $\mathcal{K}^{\mathrm{b}}(\mathrm{prj}\, \mathcal{A})$.*

The following is our main result in this paper.

Theorem 8.2. *Let* $X, X' \in \overleftarrow{\mathrm{Colax}}(I, \Bbbk\text{-}\mathbf{Cat})$. *Assume that* X *is* $\Bbbk$*-flat and that there exists a tilting colax functor* $\mathcal{T}$ *for* X *such that* $\mathcal{T}$ *and* X' *are equivalent in* $\overleftarrow{\mathrm{Colax}}(I, \Bbbk\text{-}\mathbf{Cat})$ (*the condition* (3) *in* Theorem 7.5). *Then* $\mathrm{Gr}(X)$ *and* $\mathrm{Gr}(X')$ *are derived equivalent.*

Proof. Note that $\mathrm{Gr}(X)$ is also $\Bbbk$-flat by definition of $\mathrm{Gr}(X)$. Let $\mathcal{T}$ be a tilting colax subfunctor of $\mathcal{K}^{\mathrm{b}}(\mathrm{prj}\, X)$ with an I-equivariant inclusion $(\sigma, \rho)\colon \mathcal{T} \hookrightarrow \mathcal{K}^{\mathrm{b}}(\mathrm{prj}\, X)$. Put $(P, \phi) := (P_X, \phi_X)$ for short. Let $\mathcal{T}'$ be the full subcategory of $\mathcal{K}^{\mathrm{b}}(\mathrm{prj}\,\mathrm{Gr}(X))$ consisting of the objects $\mathcal{K}^{\mathrm{b}}(\mathrm{prj}\, P(i))(U)$ with $i \in I_0$ and $U \in \mathcal{T}(i)_0$. Then $\mathcal{T}'$ is a tilting subcategory of $\mathcal{K}^{\mathrm{b}}(\mathrm{prj}\,\mathrm{Gr}(X))$. Indeed, let $L, M \in \mathcal{T}'_0$ and $0 \neq p \in \mathbb{Z}$. Then $L = \mathcal{K}^{\mathrm{b}}(\mathrm{prj}\, P(i))(U)$ and $M = \mathcal{K}^{\mathrm{b}}(\mathrm{prj}\, P(j))(V)$ for some $i, j \in I_0$ and some $U \in \mathcal{T}(i)_0$, $V \in \mathcal{T}(j)_0$. Since

$$\mathcal{K}^{\mathrm{b}}(\mathrm{prj}(P, \phi))\colon \mathcal{K}^{\mathrm{b}}(\mathrm{prj}\, X) \to \Delta(\mathcal{K}^{\mathrm{b}}(\mathrm{prj}\,\mathrm{Gr}(X)))$$

is an I-precovering by Proposition 6.8, we have

$$\begin{aligned}
\mathcal{K}^{\mathrm{b}}(\mathrm{prj}\,\mathrm{Gr}(X))(L, M[p]) &\cong \mathcal{K}^{\mathrm{b}}(\mathrm{prj}\,\mathrm{Gr}(X))(\mathcal{K}^{\mathrm{b}}(\mathrm{prj}(P, \phi))(U), \mathcal{K}^{\mathrm{b}}(\mathrm{prj}(P, \phi))(V[p])) \\
&\cong \bigoplus_{a \in I(i,j)} \mathcal{K}^{\mathrm{b}}(\mathrm{prj}\, X(j))(\mathcal{K}^{\mathrm{b}}(\mathrm{prj}\, X)(a)(U), V[p]) \\
&\overset{\text{(a)}}{\cong} \bigoplus_{a \in I(i,j)} \mathcal{K}^{\mathrm{b}}(\mathrm{prj}\, X(j))(\mathcal{T}(a)U, V[p]) \overset{\text{(b)}}{=} 0,
\end{aligned}$$

where the isomorphism (a) follows using the natural isomorphism $\rho(a)$:

$$\begin{tikzcd}
U \in \mathcal{T}(i) \arrow[r, hook] \arrow[d, "\mathcal{T}(a)"'] & \mathcal{K}^{\mathrm{b}}(\mathrm{prj}\, X(i)) \arrow[d, "\mathcal{K}^{\mathrm{b}}(\mathrm{prj}\, X)(a)"] \arrow[dl, Rightarrow, "\rho(a)"', "\cong"] \\
\mathcal{T}(i) \arrow[r, hook] & \mathcal{K}^{\mathrm{b}}(\mathrm{prj}\, X(j))
\end{tikzcd}$$

and the equality (b) follows by assumption from the fact that $\mathcal{T}(a)U, V \in \mathcal{T}(j)$. Now for a triangulated category $\mathcal{U}$ and a class of objects $\mathcal{V}$ in $\mathcal{U}$ we denote by $\mathrm{thick}\,\mathcal{V}$ the smallest thick subcategory of $\mathcal{U}$ containing $\mathcal{V}$. Then for each $i \in I_0$ and $x \in X(i)$ we have $\mathcal{K}^{\mathrm{b}}(\mathrm{prj}\, P(i))(X(i)(\text{-}, x)) \cong (\mathrm{prj}\, P(i))(X(i)(\text{-}, x)) \cong \mathrm{Gr}(X)(\text{-}, P(i)(x)) = \mathrm{Gr}(X)(\text{-}, {}_i x)$ by the formula (6.2), and hence

$$\begin{aligned}
\mathrm{Gr}(X)(\text{-}, {}_i x) &\cong \mathcal{K}^{\mathrm{b}}(\mathrm{prj}\, P(i))(X(i)(\text{-}, x)) \\
&\in \mathcal{K}^{\mathrm{b}}(\mathrm{prj}\, P(i))(\mathrm{thick}\,\mathcal{T}(i)) \\
&\subseteq \mathrm{thick}\{\mathcal{K}^{\mathrm{b}}(\mathrm{prj}\, P(i))(U) \mid U \in \mathcal{T}(i)\} \\
&\subseteq \mathrm{thick}\,\mathcal{T}'.
\end{aligned}$$

Therefore $\mathrm{thick}\,\mathcal{T}' = \mathcal{K}^{\mathrm{b}}(\mathrm{prj}\,\mathrm{Gr}(X))$, and hence $\mathcal{T}'$ is a tilting subcategory of $\mathcal{K}^{\mathrm{b}}(\mathrm{prj}\,\mathrm{Gr}(X))$, as desired. Hence $\mathrm{Gr}(X)$ and $\mathcal{T}'$ are derived equivalent because $\mathrm{Gr}(X)$ is $\Bbbk$-flat. Let (F, ψ) be the restriction of $\mathcal{K}^{\mathrm{b}}(\mathrm{prj}(P, \phi))$ to $\mathcal{T}$. Then $(F, \psi)\colon \mathcal{T} \to \Delta(\mathcal{T}')$ is a dense functor and an I-precovering, thus it is an I-covering, which shows that $\mathcal{T}' \simeq \mathrm{Gr}(\mathcal{T})$ by Corollary 5.3. Since $\mathcal{T}$ and X' are equivalent in $\overleftarrow{\mathrm{Colax}}(I, \Bbbk\text{-}\mathbf{Cat})$,

we have $\mathrm{Gr}(\mathcal{T}) \simeq \mathrm{Gr}(X')$. As a consequence, $\mathrm{Gr}(X)$ and $\mathrm{Gr}(X')$ are derived equivalent. $\square$

Corollary 8.3. *Let $X, X' \in \overleftarrow{\mathrm{Colax}}(I, \Bbbk\text{-}\mathbf{Cat})$. If X and X' are derived equivalent, then so are $\mathrm{Gr}(X)$ and $\mathrm{Gr}(X')$.*

Proof. Assume that X and X' are derived equivalent, namely that the condition (1) in Theorem 7.5 is satisfied. Then the condition (3) in Theorem 7.5 holds by Theorem 7.5 (a) and (b). Hence $\mathrm{Gr}(X)$ and $\mathrm{Gr}(X')$ are derived equivalent by the theorem above. $\square$

The following is easy to verify.

Lemma 8.4. *Let C, C' be in $\Bbbk$-**Cat**. If C and C' are derived equivalent, then so are $\Delta(C)$ and $\Delta(C')$.* $\square$

Corollary 8.3 together with the lemma above and Example 4.2 gives us a unified proof of the following fact.

Theorem 8.5. *Assume that $\Bbbk$ is a field and that $\Bbbk$-algebras A and A' are derived equivalent. Then the following pairs are derived equivalent as well:*

(1) *path-categories AQ and $A'Q$ for any quiver Q;*
(2) *incidence categories AS and $A'S$ for any poset S; and*
(3) *monoid algebras AG and $A'G$ for any monoid G.*

$\square$

Example 8.6. Assume that $\Bbbk$ is a field. Let n be a natural number ≥ 3, and I the free category defined by the quiver Q: $2 \xrightarrow{a_2} 3 \xrightarrow{a_3} \cdots \xrightarrow{a_{n-1}} n$. Define functors $X, X'\colon I \to \Bbbk$-**Cat** as follows.

For each $i \in I_0 = \{2, \dots, n\}$ let $X(i)$ be the $\Bbbk$-category defined by the quiver

$$\begin{tikzcd} 1 \arrow[r, bend left, "\alpha_1"] & 2 \arrow[l, bend left, "\beta_1"] \arrow[r, bend left, "\alpha_2"] & 3 \arrow[l, bend left, "\beta_2"] \arrow[r, bend left, "\alpha_3"] & \cdots \arrow[l, bend left, "\beta_3"] \arrow[r, bend left, "\alpha_{i-1}"] & i \arrow[l, bend left, "\beta_{i-1}"] \end{tikzcd}$$

with relations $\alpha_{j+1}\alpha_j = 0$, $\beta_j\beta_{j+1} = 0$, $\alpha_j\beta_j = \beta_{j+1}\alpha_{j+1}$ for all $j = 1, \dots, i-1$ and $\alpha_1\beta_1\alpha_1 = 0$, $\beta_{i-1}\alpha_{i-1}\beta_{i-1} = 0$. For each $a_i\colon i \to i+1$ in I_1 let $X(a_i)\colon X(i) \to X(i+1)$ be the inclusion functor. This defines a functor $X\colon I \to \Bbbk$-**Cat**.

For each $i \in I_0 = \{2, \dots, n\}$ let $X'(i)$ be the $\Bbbk$-category defined by the quiver

$$\begin{tikzcd} 1 \arrow[r, "\gamma_1"] & 2 \arrow[r, "\gamma_2"] & 3 \arrow[r, "\gamma_3"] & \cdots \arrow[r, "\gamma_{i-1}"] & i \arrow[llll, bend left, "\gamma_i"] \end{tikzcd}$$

with relations $\gamma_{j+i} \cdots \gamma_{j+1}\gamma_j = 0$ for all $j \in \mathbb{Z}/i\mathbb{Z}$. For each $a_i\colon i \to i+1$ in I_1 let $X(a_i)\colon X(i) \to X(i+1)$ be the functor defined by the correspondence $1 \mapsto 1$, $j \mapsto j+1$ and $\alpha_1 \mapsto \alpha_2\alpha_1$, $\alpha_j \mapsto \alpha_{j+1}$ for all $j = 2, \dots i$. This defines a functor $X'\colon I \to \Bbbk$-**Cat**.

As is explained in [1] we have a tilting spectroid $\mathcal{T}(i)$ for $X(i)$ that is a full subcategory of $\mathcal{K}^{\mathrm{b}}(\operatorname{prj} X(i))$ consisting of the following i objects

$$
\begin{aligned}
T(i)_1 &:= (\underline{P_1}),\\
T(i)_2 &:= (\underline{P_2} \xrightarrow{P(\alpha_2)} P_3 \xrightarrow{P(\alpha_3)} \cdots \xrightarrow{P(\alpha_{i-1})} P_i),\\
T(i)_3 &:= (\underline{P_2} \xrightarrow{P(\alpha_2)} P_3 \xrightarrow{P(\alpha_3)} \cdots \xrightarrow{P(\alpha_{i-2})} P_{i-1}),\\
&\ \vdots\\
T(i)_i &:= (\underline{P_2}),
\end{aligned}
$$

where $P_j := X(i)(\text{-}, j) \in \operatorname{prj} X(i)$ for all $j \in X(i)_0$, $P(\alpha) := X(i)(\text{-}, \alpha)$ for all $\alpha \in X(i)_1$ and the underline indicates the place of degree zero. Again by [1], $\mathcal{T}(i)$ is presented by the same quiver with relations as $X'(i)$ and we have an isomorphism $F(i)\colon X'(i) \to \mathcal{T}(i)$ sending j to $T(i)_j$ for all $j = 1, \dots, i$ and γ_j to a morphism $\delta(i)_j\colon T(i)_j \to T(i)_{j+1}$ for all $j \in \mathbb{Z}/i\mathbb{Z}$, where $\delta(i)_1 := (\underline{P(\alpha_1)})$, $\delta(i)_j := (\underline{\mathbb{1}_{P_2}}, \dots, \mathbb{1}_{P_{i-j+1}}, 0)$ for all $j = 2, \dots, i-1$ and $\delta(i)_i := (\underline{P(\beta_1)})$. Thus $\mathcal{T}(i)$ gives a derived equivalence between $X(i)$ and $X'(i)$.

For each $a_i\colon i \to i+1$ in I_1 define a functor $\mathcal{T}(a_i)\colon \mathcal{T}(i) \to \mathcal{T}(i+1)$ by the correspondence $T(i)_1 \mapsto T(i+1)_1$, $T(i)_j \mapsto T(i+1)_{j+1}$ and $\delta(i)_1 \mapsto \delta(i+1)_2\delta(i+1)_1$, $\delta(i)_j \mapsto \delta(i+1)_{j+1}$ for all $j = 2, \dots, i$. This defines a functor $\mathcal{T}\colon I \to \mathbb{k}\text{-}\mathbf{Cat}$. Then we have a strict commutative diagram

$$
\begin{tikzcd}
X'(i) \arrow[r, "F(i)"] \arrow[d, "X'(a_i)"'] & \mathcal{T}(i) \arrow[d, "\mathcal{T}(a_i)"] \\
X'(i+1) \arrow[r, "F(i+1)"'] & \mathcal{T}(i+1)
\end{tikzcd}
$$

in $\mathbb{k}\text{-}\mathbf{Cat}$ for all $i \in I_0$, which shows that X' and $\mathcal{T}$ are equivalent in $\overleftarrow{\mathrm{Colax}}(I, \mathbb{k}\text{-}\mathbf{Cat})$. Finally by definition of $\mathcal{T}(a_i)$'s it is easy to see that we have an I-equivariant inclusion $(\sigma, \rho)\colon \mathcal{T} \hookrightarrow \mathcal{K}^{\mathrm{b}}(\operatorname{prj} X)$:

$$
\begin{tikzcd}
\mathcal{T}(i) \arrow[r, hook, "\sigma(i)"] \arrow[d, "\mathcal{T}(a_i)"'] & \mathcal{K}^{\mathrm{b}}(\operatorname{prj} X(i)) \arrow[d, "\mathcal{K}^{\mathrm{b}}(\operatorname{prj} X(a_i))"] \arrow[dl, Rightarrow, "\rho(a_i)"', "\cong"] \\
\mathcal{T}(i+1) \arrow[r, hook, "\sigma(i+1)"'] & \mathcal{K}^{\mathrm{b}}(\operatorname{prj} X(i+1)).
\end{tikzcd}
$$

Hence by Theorem 8.2 we can glue derived equivalences between $X(i)$'s and $X'(i)$'s together to have a derived equivalence between $\mathrm{Gr}(X)$ and $\mathrm{Gr}(X')$. For example when

$n = 5$, these are presented by the following quivers

$$
\mathrm{Gr}(X) = \begin{tikzcd}
1 \arrow[r, bend left, "\alpha_1"] \arrow[d] & 2 \arrow[l, bend left, "\beta_1"] \arrow[d] \\
1 \arrow[r, bend left, "\alpha_1"] \arrow[d] & 2 \arrow[l, bend left, "\beta_1"] \arrow[r, bend left, "\alpha_2"] \arrow[d] & 3 \arrow[l, bend left, "\beta_2"] \arrow[d] \\
1 \arrow[r, bend left, "\alpha_1"] \arrow[d] & 2 \arrow[l, bend left, "\beta_1"] \arrow[r, bend left, "\alpha_2"] \arrow[d] & 3 \arrow[l, bend left, "\beta_2"] \arrow[r, bend left, "\alpha_3"] \arrow[d] & 4 \arrow[l, bend left, "\beta_3"] \arrow[d] \\
1 \arrow[r, bend left, "\alpha_1"] & 2 \arrow[l, bend left, "\beta_1"] \arrow[r, bend left, "\alpha_2"] & 3 \arrow[l, bend left, "\beta_2"] \arrow[r, bend left, "\alpha_3"] & 4 \arrow[l, bend left, "\beta_3"] \arrow[r, bend left, "\alpha_4"] & 5 \arrow[l, bend left, "\beta_4"]
\end{tikzcd}
, \quad
\mathrm{Gr}(X') = \begin{tikzcd}
1 \arrow[r, "\gamma_1"] \arrow[d] & 2 \arrow[l, bend left, "\gamma_2"] \arrow[dr] \\
1 \arrow[r, "\gamma_1"] \arrow[d] & 2 \arrow[r, "\gamma_2"] \arrow[dr] & 3 \arrow[ll, bend left, "\gamma_3"] \arrow[dr] \\
1 \arrow[r, "\gamma_1"] \arrow[d] & 2 \arrow[r, "\gamma_2"] \arrow[dr] & 3 \arrow[r, "\gamma_3"] \arrow[dr] & 4 \arrow[lll, bend left, "\gamma_4"] \arrow[dr] \\
1 \arrow[r, "\gamma_1"] & 2 \arrow[r, "\gamma_2"] & 3 \arrow[r, "\gamma_3"] & 4 \arrow[r, "\gamma_4"] & 5 \arrow[llll, bend left, "\gamma_5"]
\end{tikzcd}
$$

with suitable relations as calculated in [7]. Note that if we start with I presented by the same quiver Q as above with relations $a_{i+1}a_i = 0$ for all $i = 2, \dots, n-2$, then both $\mathrm{Gr}(X)$ and $\mathrm{Gr}(X')$ are presented by the same quivers with relations consisting of the same relations as before together with the additional relations that the vertical paths of length 2 are zero, respectively.

9. The composite of colax functors and pseudofunctors

In this section we prove Theorem 6.5. Throughout this section $\mathbf{B}, \mathbf{C}$ and $\mathbf{D}$ are 2-categories.

Notation 9.1. When we denote a colax functor by a letter X the 1-st (resp. 2-nd and 3-rd) entry of X is denoted by $X_{012} := (X_0, X_1, X_2)$ (resp. η^X and θ^X), thus we set $X = (X_{012}, \eta^X, \theta^X)$, and sometimes we simply write X for X_d for all $d = 0, 1, 2$ if this seems to make no confusion.

9.1. Correspondences on cells.

Lemma 9.2. *Let $X\colon \mathbf{B} \to \mathbf{C}$ and $V\colon \mathbf{C} \to \mathbf{D}$ be colax functors. We define the composite $VX\colon \mathbf{B} \to \mathbf{D}$ as follows.*

- $(VX)_d := V_d X_d\colon \mathbf{B}_d \xrightarrow{X_d} \mathbf{C}_d \xrightarrow{V_d} \mathbf{D}_d$ *for all* $d = 0, 1, 2$.
- $\eta_i^{VX} := \eta^V_{X(i)} \circ V\eta_i^X\colon\ VX(\mathbb{1}_i) \xRightarrow{V\eta_i^X} V(\mathbb{1}_{X(i)}) \xRightarrow{\eta^V_{X(i)}} \mathbb{1}_{(VX)(i)}$ *for all* $i \in \mathbf{B}_0$.
- $\theta_{b,a}^{VX} := \theta^V_{X(b),X(a)} \circ V\theta^X_{b,a}\colon\ VX(ba) \xRightarrow{V\theta^X_{b,a}} V(X(b) \circ X(a)) \xRightarrow{\theta^V_{X(b),X(a)}} VX(b) \circ VX(a)$ *for all* $(b, a) \in \mathrm{com}(\mathbf{B})$.

Namely, $VX := ((V_0X_0, V_1X_1, V_2X_2), (\eta^V_{X(i)} \circ V\eta_i^X)_{i\in\mathbf{B}_0}, (\theta^V_{X(b),X(a)} \circ V\theta^X_{b,a})_{(b,a)\in\mathrm{com}(\mathbf{B})})$. *Then the composite* $\overleftarrow{\mathrm{Colax}}(\mathbf{B}, V)(X) := VX\colon \mathbf{B} \to \mathbf{D}$ *is again a colax functor.*

Proof. It is enough to verify the axioms (i) – (iv) in Definition 6.1.

(i) $((VX)_1, (VX)_2)\colon \mathbf{B}(i, j) \xrightarrow{(X_1,X_2)} \mathbf{C}(X(i), X(j)) \xrightarrow{(V_1,V_2)} \mathbf{D}(VX(i), VX(j))$ is a functor for all $i, j \in \mathbf{B}_0$ as a composite of the functors (X_1, X_2) and (V_1, V_2).

(ii) For each $a\colon i \to j$ in $\mathbf{B}$ we have the following commutative diagram:

$$\begin{tikzcd}
VX(a)\mathbb{1}_{VX(i)} \arrow[dr, equal] & VX(a)V(\mathbb{1}_{X(i)}) \arrow[l, Rightarrow, "VX(a)\eta^V_{X(i)}"'] & VX(a)VX(\mathbb{1}_i) \arrow[l, Rightarrow, "VX(a)V(\eta^X_i)"'] \\
& V(X(a)\mathbb{1}_{X(i)}) \arrow[u, Rightarrow, "\theta^V_{X(a),\mathbf{1}_{X(i)}}"'] \arrow[dr, equal] & V(X(a)X(\mathbb{1}_i)) \arrow[l, Rightarrow, "V(X(a)\eta^X_i)"'] \arrow[u, Rightarrow, "\theta^V_{X(a),X(\mathbf{1}_i)}"'] \\
& & VX(a\mathbb{1}_i). \arrow[u, Rightarrow, "V(\theta^X_{a,\mathbf{1}_i})"']
\end{tikzcd}$$

The commutativity of the square follows from the axiom (iv) for θ^V. The remaining commutative diagram is obtained similarly. These two commutative diagrams verify the axiom (ii) of colax functors.

(iii) For each $i \xrightarrow{a} j \xrightarrow{b} k \xrightarrow{c} l$ in $\mathbf{B}$ we have the following commutative diagram:

$$\begin{tikzcd}
VX(cba) \arrow[r, Rightarrow, "V\theta^X_{c,ba}"] \arrow[d, Rightarrow, "V\theta^X_{cb,a}"'] & V(X(c)X(ba)) \arrow[r, Rightarrow, "\theta^V_{X(c),X(ba)}"] \arrow[d, Rightarrow, "V(\mathbf{1}_{X(c)}\theta^X_{b,a})"] & VX(c)\cdot VX(ba) \arrow[d, Rightarrow, "VX(c)\cdot V\theta^X_{b,a}"] \\
V(X(cb)X(a)) \arrow[r, Rightarrow, "V(\theta^X_{c,b}\mathbf{1}_{X(a)})"] \arrow[d, Rightarrow, "\theta^V_{X(cb),X(a)}"'] & V(X(c)X(b)X(a)) \arrow[r, Rightarrow, "\theta^V_{X(c),X(b)X(a)}"] \arrow[d, Rightarrow, "\theta^V_{X(c)X(b),X(a)}"] & VX(c)V(X(b)X(a)) \arrow[d, Rightarrow, "VX(c)\theta^V_{X(b),X(a)}"] \\
VX(cb)\cdot VX(a) \arrow[r, Rightarrow, "V(\theta^X_{c,b})VX(a)"'] & V(X(c)X(b))VX(a) \arrow[r, Rightarrow, "\theta^V_{X(c),X(b)}VX(a)"'] & VX(c)\cdot VX(b)\cdot VX(a),
\end{tikzcd}$$

which verifies the axiom (iii) of colax functors.

(iv) Let $a, a'\colon i \to j$; $b, b'\colon j \to k$; $\alpha\colon a \Rightarrow a'$ and $\beta\colon b \Rightarrow b'$ be in $\mathbf{B}$. Then we have the following commutative diagram:

$$\begin{tikzcd}
VX(ba) \arrow[r, Rightarrow, "V(\theta^X_{b,a})"] \arrow[d, Rightarrow, "VX(\beta\cdot\alpha)"'] & V(X(b)\cdot X(a)) \arrow[r, Rightarrow, "\theta^V_{X(b),X(a)}"] \arrow[d, Rightarrow, "V(X\beta\cdot X\alpha)"] & VX(b)\cdot VX(a) \arrow[d, Rightarrow, "VX\beta\cdot VX\alpha"] \\
VX(b'a') \arrow[r, Rightarrow, "V(\theta^X_{b',a'})"'] & V(X(b')\cdot V(a')) \arrow[r, Rightarrow, "\theta^V_{X(b'),X(a')}"'] & VX(b')\cdot VX(a'),
\end{tikzcd}$$

which verifies the axiom (iv) of colax functors. $\square$

Lemma 9.3. *Let $X, X' \colon \mathbf{B} \to \mathbf{C}$ and $V \colon \mathbf{C} \to \mathbf{D}$ be colax functors and $(F, \psi) \colon X \to X'$ a 1-morphism in $\overleftarrow{\mathrm{Colax}}(\mathbf{B}, \mathbf{C})$, and consider the diagram*

$$
\begin{tikzcd}[column sep=huge, row sep=large]
VX(i) \arrow[rr, "VF(i)"] \arrow[dd, "VX(a)"'] \arrow[dr, phantom, "V(X'(a)F(i))"] \arrow[ddrr, bend left] \arrow[ddrr, bend right] & & VX'(i) \arrow[dd, "VX'(a)"] \\
& {\Downarrow V\psi(a)} \quad V(F(j)X(a)) & \\
VX(j) \arrow[rr, "VF(j)"'] & & VX'(j).
\end{tikzcd}
\tag{9.3}
$$

$\theta^V_{X'(a),F(i)} \colon V(X'(a)F(i)) \Rightarrow VX'(a)\cdot VF(i)$, $\quad \theta^V_{F(j),X(a)} \colon V(F(j)X(a)) \Rightarrow VF(j)\cdot VX(a)$

Assume that $\theta^V_{d,c}$ are isomorphisms for all $(d, c) \in \mathrm{com}(\mathbf{C})$ (e.g., that V is a pseudofunctor). Then we can define a 1-morphism $\overleftarrow{\mathrm{Colax}}(\mathbf{B}, V)(F, \psi) := V(F, \psi) \colon VX \to VX'$ in $\overleftarrow{\mathrm{Colax}}(\mathbf{B}, \mathbf{D})$ by

$$
\begin{aligned}
V(F, \psi) &:= ((V(F(i)))_{i \in \mathbf{B}_0}, (\psi_V(a))_{a \in \mathbf{B}_1}), \ \text{where for } a : i \to j \\
\psi_V(a) &:= \theta^V_{F(j),X(a)} \cdot V(\psi(a)) \cdot {\theta^V_{X'(a),F(i)}}^{-1}.
\end{aligned}
$$

Proof. We set $X = (X, \eta, \theta)$ and $X' = (X', \eta', \theta')$ for short.

First, the functor $V_{12} \colon \mathbf{C}(X(i), X'(j)) \to \mathbf{D}(VX(i), VX'(j))$ sends the commutative square (6.1) to the commutative square $(*)$ below

$$
\begin{tikzcd}[column sep=large]
VX'(a)\cdot VF(i) \arrow[rrr, Rightarrow, "VX'(\alpha)*VF(i)"] \arrow[ddd, Rightarrow, "\psi_V(a)"'] \arrow[dr, phantom, "\text{(definition)}", near end] & & & VX'(b)\cdot VF(i) \arrow[ddd, Rightarrow, "\psi_V(b)"] \\
& V(X'(a)F(i)) \arrow[r, Rightarrow, "V(X'(\alpha)F(i))"] \arrow[ul, Rightarrow, "\theta^V_{X'(a),F(i)}"'] \arrow[d, Rightarrow, "V(\psi(a))"'] \arrow[u, phantom, "\text{(iv)}"] \arrow[dr, phantom, "(*)"] & V(X'(b)F(i)) \arrow[ur, Rightarrow, "\theta^V_{X'(b),F(i)}"] \arrow[d, Rightarrow, "V(\psi(b))"] \arrow[dr, phantom, "\text{(definition)}"] & \\
& V(F(j)X(a)) \arrow[r, Rightarrow, "V(F(j)X(\alpha))"'] \arrow[dl, Rightarrow, "\theta^V_{F(j),X(a)}"] \arrow[d, phantom, "\text{(iv)}"] & V(F(j)X(b)) \arrow[dr, Rightarrow, "\theta^V_{F(j),X(b)}"] & \\
VF(j)\cdot VX(a) \arrow[rrr, Rightarrow, "VF(j)*VX(\alpha)"'] & & & VF(j)\cdot VX(b),
\end{tikzcd}
$$

which is completed to the commutative diagram above. Hence the family $(\psi_V(a))_{a \in \mathbf{B}_1}$ has the property (0) of 1-morphisms in $\overleftarrow{\mathrm{Colax}}(\mathbf{B}, \mathbf{D})$ (Definition 6.1(4)).

(a) For each $i \in \mathbf{B}_0$ we have the following commutative diagram:

$$
\begin{tikzcd}[column sep=large]
VX'(\mathbb{1}_i)\cdot VF(i) \arrow[d, Rightarrow, "V\eta'_i\cdot VF(i)"] & V(X'(\mathbb{1}_i)\cdot F(i)) \arrow[l, Rightarrow, "\theta^V_{X'(\mathbb{1}_i),F(i)}"'] \arrow[r, Rightarrow, "V\psi(\mathbb{1}_i)"] \arrow[d, Rightarrow, "V(\eta'_i \mathbf{1}_{F(i)})"] & V(F(i)\cdot X(\mathbb{1}_i)) \arrow[r, Rightarrow, "\theta^V_{F(i),X(\mathbb{1}_i)}"] \arrow[d, Rightarrow, "V(\mathbf{1}_{F(i)}\eta_i)"] & VF(i)\cdot VX(\mathbb{1}_i) \arrow[d, Rightarrow, "VF(i)\cdot V\eta_i"'] \\
V(\mathbb{1}_{X'(i)})\cdot VF(i) \arrow[d, Rightarrow, "\eta^V_{X'(i)}\cdot VF(i)"] & V(\mathbb{1}_{X'(i)}\cdot F(i)) \arrow[l, Rightarrow, "\theta^V_{\mathbb{1}_{X'(i)},F(i)}"] \arrow[r, equal] & V(F(i)\cdot \mathbb{1}_{X(i)}) \arrow[r, Rightarrow, "\theta^V_{F(i),\mathbb{1}_{X(i)}}"'] & VF(i)\cdot V(\mathbb{1}_{X(i)}) \arrow[d, Rightarrow, "VF(i)\cdot\eta^V_{X(i)}"'] \\
\mathbb{1}_{VX'(i)}\cdot VF(i) \arrow[rrr, equal] & & & VF(i)\cdot \mathbb{1}_{VX(i)},
\end{tikzcd}
$$

which verifies the axiom (a) of 1-morphisms.

(b) For each $i \xrightarrow{a} j \xrightarrow{b} k$ in $\mathbf{B}$ we have the following commutative diagrams:

$$\begin{tikzcd}
VX'(ba)\cdot VF(i) \arrow[r, Rightarrow, "V(\theta'_{b,a})VF(i)"] & V(X'(b)X'(a))\cdot VF(i) \arrow[r, Rightarrow, "\theta^V_{X'(b),X'(a)}VF(i)"] & VX'(b)\cdot VX'(a)\cdot VF(i) \arrow[d, Rightarrow, "VX'(b)\theta^V_{X'(a),F(i)}{}^{-1}"'] & \\
 & & VX'(b)V(X'(a)F(i)) \arrow[r, Rightarrow, "V(\mathbf{1}_{X'(b)})V(\psi(a))"] & VX'(b)\cdot V(F(j)X(a)) \arrow[d, Rightarrow, "VX'(b)\cdot\theta^V_{F(j),X(a)},"] \\
 & & & VX'(b)\cdot VF(j)\cdot VX(a) \arrow[d, Rightarrow, "\theta^V_{X'(b),F(j)}{}^{-1}VX(a)"] \\
V(X'(ba)F(i)) \arrow[uuu, Rightarrow, "\theta^V_{X'(ba),F(i)}"] \arrow[r, Rightarrow, "V(\theta'_{b,a}F(i))"] & V(X'(b)X'(a)F(i)) \arrow[uuu, Rightarrow, "\theta^V_{X'(b)X'(a),F(i)}"] \arrow[uur, Rightarrow, "\theta^V_{X(b),X'(a)F(i)}"] \arrow[r, Rightarrow, "V(X'(b)\psi(a))"] & V(X'(b)F(j)X(a)) \arrow[uur, Rightarrow, "\theta^V_{X'(b),F(j)X(a)}"] \arrow[r, Rightarrow, "\theta^V_{X'(b)F'(j),X(a)}"] & V(X'(b)F(j))VX(a)
\end{tikzcd}$$

and

$$\begin{tikzcd}
V(X'(ba)F(i)) \arrow[r, Rightarrow, "V(\theta'_{b,a}F(i))"] \arrow[d, Rightarrow, "V\psi(ba)"'] & V(X'(b)X'(a)F(i)) \arrow[r, Rightarrow, "V(X'(b)\psi(a))"] & V(X'(b)F(j)X(a)) \arrow[r, Rightarrow, "\theta^V_{X'(b)F'(j),X(a)}"] \arrow[d, Rightarrow, "V(\psi(b)X(a))"'] & V(X'(b)F(j))VX(a) \arrow[d, Rightarrow, "V(\psi(b))VX(a)"] \\
V(F(k)X(ba)) \arrow[rr, Rightarrow, "V(F(k)\theta_{b,a})"] \arrow[d, Rightarrow, "\theta^V_{F(k),X(ba)}"'] & & V(F(k)X(b)X(a)) \arrow[r, Rightarrow, "\theta^V_{X(b)F'(j),X(a)}"] \arrow[d, Rightarrow, "\theta^V_{F(k),X(b)X(a)}"'] & V(F(k)X(b))VX(a) \arrow[d, Rightarrow, "\theta^V_{F(k),X(b)}VX(a)"] \\
VF(k)\cdot VX(ba) \arrow[rr, Rightarrow, "VF(k)\cdot V\theta_{b,a}"'] & & VF(k)\cdot V(X(b)X(a)) \arrow[r, Rightarrow, "VF(k)\cdot\theta^V_{X(b),X(a)}"'] & VF(k)\cdot VX(b)\cdot VX(a).
\end{tikzcd}$$

Glue these two diagrams together along the common row to get a large diagram, which verifies the axiom (b) of 1-morphisms. $\square$

Lemma 9.4. *Let $X, X' \colon \mathbf{B} \to \mathbf{C}$ and $V \colon \mathbf{C} \to \mathbf{D}$ be colax functors, $(F,\psi), (F',\psi') \colon X \to X'$ 1-morphisms, and $\alpha \colon (F,\psi) \Rightarrow (F',\psi')$ a 2-morphism in $\overleftarrow{\mathrm{Colax}}(\mathbf{B},\mathbf{C})$. Assume that all $\theta^V_{d,c}$ are isomorphisms (e.g., that V is a pseudofunctor). Then we can define a 2-morphism $\overleftarrow{\mathrm{Colax}}(\mathbf{B},V)(\alpha) := V\alpha \colon V(F,\psi) \Rightarrow V(F',\psi')$ in $\overleftarrow{\mathrm{Colax}}(\mathbf{B},\mathbf{D})$ by*

$$V\alpha := (V\alpha_i)_{i\in\mathbf{B}_0}.$$

Proof. Let $a \colon i \to j$ be in $\mathbf{B}$. It is enough to show the commutativity of the following diagram:

$$\begin{tikzcd}
VX'(a)\cdot VF(i) \arrow[d, Rightarrow, "VX'(a)\cdot V\alpha_i"'] & V(X'(a)F(i)) \arrow[l, Rightarrow, "\theta^V_{X'(a),F(i)}"'] \arrow[r, Rightarrow, "V(\psi(a))"] \arrow[d, Rightarrow, "V(X'(a)\alpha_i)"'] & V(F(j)X(a)) \arrow[r, Rightarrow, "\theta^V_{F(j),X(a)}"] \arrow[d, Rightarrow, "V(\alpha_j X(a))"] & VF(j)\cdot VX(a) \arrow[d, Rightarrow, "V\alpha_i\cdot VX(a)"] \\
VX'(a)\cdot VF'(i) & V(X'(a)F'(i)) \arrow[l, Rightarrow, "\theta^V_{X'(a),F'(i)}"] \arrow[r, Rightarrow, "V(\psi'(a))"'] & V(F'(j)X(a)) \arrow[r, Rightarrow, "\theta^V_{F'(j),X(a)}"'] & VF'(j)\cdot VX(a).
\end{tikzcd}$$

Since $\alpha = (\alpha_i\colon F(i) \Rightarrow F'(i))_{i\in\mathbf{B}_0}$ is a 2-morphism in $\overleftarrow{\mathrm{Colax}}(\mathbf{B},\mathbf{C})$, we have the commutative diagram

$$\begin{tikzcd}
X'(a)F(i) \arrow[r, Rightarrow, "\psi(a)"] \arrow[d, Rightarrow, "X'(a)\alpha_i"'] & F(j)X(a) \arrow[d, Rightarrow, "\alpha_j X(a)"] \\
X'(a)F'(i) \arrow[r, Rightarrow, "\psi'(a)"'] & F'(j)X(a).
\end{tikzcd}$$

This gives the commutativity of the central square of the diagram above by applying the functor (V_1, V_2) to it. The axiom (iv) of colax functors for V shows the commutativity of the remaining squares. $\square$

9.2. **Proof of Theorem 6.5.** By the three lemmas above we can define a correspondence

$$\overleftarrow{\mathrm{Colax}}(\mathbf{B},V)_{012}\colon \overleftarrow{\mathrm{Colax}}(\mathbf{B},\mathbf{C}) \to \overleftarrow{\mathrm{Colax}}(\mathbf{B},\mathbf{D})$$

sending i-cells to i-cells for all $i = 0,1,2$ preserving domains and codomains. It remains to define families $H = (H_X)_{X\in\overleftarrow{\mathrm{Colax}}(\mathbf{B},\mathbf{C})_0}$ and $\Theta = (\Theta_{F',F})_{(F',F)\in\mathrm{com}(\overleftarrow{\mathrm{Colax}}(\mathbf{B},\mathbf{C}))}$ and to show that $\overleftarrow{\mathrm{Colax}}(\mathbf{B},V) := (\overleftarrow{\mathrm{Colax}}(\mathbf{B},V)_{012}, H, \Theta)$ becomes a pseudofunctor $\overleftarrow{\mathrm{Colax}}(\mathbf{B},\mathbf{C}) \to \overleftarrow{\mathrm{Colax}}(\mathbf{B},\mathbf{D})$.

For each $X \in \overleftarrow{\mathrm{Colax}}(\mathbf{B},\mathbf{C})_0$ we define $H_X\colon V(\mathbb{1}_X) \Rightarrow \mathbb{1}_{VX}$ by setting

$$H_X := (\eta^V_{X(i)}\colon V(\mathbb{1}_{X(i)}) \to \mathbb{1}_{VX(i)})_{i\in\mathbf{B}_0}.$$

Then H_X turns out to be a 2-morphism because by definitions of θ^V and η^V we have a commutative diagram

$$\begin{tikzcd}
VX(a)\cdot V(\mathbb{1}_{X(i)}) \arrow[r, Rightarrow, "(\theta^V_{X(a),\mathbb{1}_{X(i)}})^{-1}"] \arrow[d, Rightarrow, "VX(a)\cdot\eta^V_{X(i)}"] & V(X(a)\cdot\mathbb{1}_{X(i)}) \arrow[r, equal] & V(\mathbb{1}_{X(j)}X(a)) \arrow[r, Rightarrow, "\theta^V_{\mathbb{1}_{X(j)},X(a)}"] & V(\mathbb{1}_{X(j)})\cdot VX(a) \arrow[d, Rightarrow, "\eta^V_{X(j)}\cdot VX(a)"'] \\
VX(a)\cdot\mathbb{1}_{VX(i)} \arrow[rrr, equal] & & & \mathbb{1}_{VX(j)}VX(a)
\end{tikzcd}$$

for all $a\colon i\to j$ in $\mathbf{B}$. Note that H_X are isomorphisms because η^V_k are for all $k\in\mathbf{C}_0$.

For each $(F',F)\in\mathrm{com}(\overleftarrow{\mathrm{Colax}}(\mathbf{B},\mathbf{C}))$, say $F\colon X\Rightarrow X'$ and $F'\colon X'\Rightarrow X''$, we define $\Theta_{F',F}\colon V(F'F)\Rightarrow VF'\circ VF$ by setting

$$\Theta_{F',F} := (\theta^V_{F'(i),F(i)}\colon V(F'(i)F(i)) \to VF'(i)\cdot VF(i))_{i\in\mathbf{B}_0}.$$

Then $\Theta_{F',F}$ turns out to be a 2-morphism. Indeed, it is enough to show the commutativity of the diagram

$$\begin{tikzcd}
VX''(a)\cdot V(F'(i)F(i)) \arrow[r, Rightarrow, "\Psi(a)"] \arrow[d, Rightarrow, "VX''(a)\cdot\theta^V_{F'(i),F(i)}"'] & V(F'(j)F(j))\cdot VX(a) \arrow[d, Rightarrow, "\theta^V_{F'(j),F(j)}\cdot VX(a)"] \\
VX''(a)\cdot VF'(i)\cdot VF(i) \arrow[r, Rightarrow, "\Psi'(a)"'] & VF'(j)VF(j)VX(a)
\end{tikzcd}$$

for all $a\colon i \to j$ in $\mathbf{B}$, where we set $V(F'F) = ((V(F'(i)F(i))_{i\in\mathbf{B}_0}, (\Psi(a))_{a\in\mathbf{B}_1})$ and $VF' \cdot VF = ((VF(i) \cdot VF(i))_{i\in\mathbf{B}_0}, (\Psi'(a))_{a\in\mathbf{B}_1})$, namely

$$\Psi(a) := \theta^V_{F'(j)F(j),X(a)} \cdot V((F'(j) \cdot \psi(a)) \cdot V(\psi'(a) \cdot F(i)) \cdot ({\theta'}^V_{X''(a),F'(i)F(i)})^{-1}$$

$$\Psi'(a) := (VF'(j) \cdot (\theta^V_{F(j),X(a)} \cdot V\psi(a) \cdot ({\theta'}^V_{X'(a),F(i)})^{-1}) \circ (\theta^V_{F'(j),X'(a)} \cdot V\psi'(a) \cdot ({\theta'}^V_{X''(a),F'(i)})^{-1} \cdot VF(i))$$

for all $a\colon i \to j$ in $\mathbf{B}$. This follows from the coassociativity of V and the naturality of θ^V. Note that $\Theta_{F',F}$ are isomorphisms because $\theta^V_{b,a}$ are for all $a, b \in \mathbf{C}_0$.

Now the defining conditions of θ^V and η^V directly show that $(\overleftarrow{\mathrm{Colax}}(\mathbf{B}, V)_{012}, H, \Theta)$ is a colax functor, hence a pseudofunctor because all H_X and $\Theta_{F',F}$ are isomorphisms. $\square$

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